Competitive versus collusive Nash equilibria in affine supply function competition with quadratic costs

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Abstract

This paper studies strategic games of supply function competition under the assumptions that (i) the sellers can choose arbitrary increasing or decreasing affine supply functions, (ii) they have an identical quadratic cost function, and (iii) the deterministic affine demand function is strictly decreasing. Based on the description of all symmetric Nash equilibria—and the ensuing characterization of an extreme-point price function as a Möbius transformation—, I derive an 'almost anything goes' result which determines the set of all market-clearing prices that can be generated by the symmetric Nash equilibria. I discuss in detail competitive Nash equilibria—which generate the minimal equilibrium price at zero profits for the sellers—and collusive Nash equilibria—which generate the monopoly price. Both extreme types of Nash equilibria uniquely exist for non-zero quadratic costs and consist of strictly decreasing supply functions.

Keywords: Tacit collusion; Möbius transformation
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1 Introduction

In static games of supply function competition strategic sellers submit supply functions to a Walrasian auctioneer who determines the market-clearing price at which the aggregate supply meets a non-strategic buyer's demand. In contrast to competitive equilibrium models the sellers are not price-takers but try to influence the market price through their choice of supply functions. If there is enough flexibility in the set of admissible supply functions, many different Nash equilibria might exist which give rise to a multitude of possible economic outcomes.

This paper comprehensively characterizes all symmetric Nash equilibria—and the corresponding equilibrium prices—for static games of *affine* supply function competition in which all sellers have identical quadratic (or zero) cost functions and face a strictly decreasing affine demand function. The following main results obtain for non-zero quadratic costs.

1. Theorem 1 derives all symmetric Nash equilibria under the assumption that the sellers' supply functions can have arbitrary intercepts and slopes. The main formal challenge of Theorem 1 is the identification of slopes which satisfy the second-order condition for an utility maximum (=SOC). There exist two separated families of symmetric Nash equilibria which satisfy the SOC: Nash equilibria in supply functions with increasing and moderately decreasing slopes, on the one hand, and Nash equilibria in supply functions with steeply decreasing slopes, on the other hand.

2. I show that the 'extreme-point' price function—which generates the prices for symmetric strategy-profiles that satisfy the first-order condition for either an utility maximum or minimum (or both)—has the mathematical structure of a *Möbius transformation* in the slope of the supply function.

3. Based on the properties of this Möbius transformation, Proposition 1 describes the set of all market-clearing prices that can be generated by some symmetric Nash equilibrium as the image of the subset of slopes satisfying the SOC for an utility maximum under the extreme-point price function. This yields an 'almost anything goes' result according to which all prices—with the exception of exactly two

prices—that come with a nonnegative profit for the sellers can be generated by some symmetric Nash equilibrium.

4. Proposition 2 establishes the existence of an unique 'competitive' Nash equilibrium which generates the minimal equilibrium price at which the sellers make zero profits.

5. Proposition 3 derives the unique 'collusive' Nash equilibrium that generates the monopoly price at which the sellers' aggregate profit is maximized.

As an important qualitative insight, competitive as well collusive Nash equilibria only exist for the non-zero quadratic cost model if the sellers can choose strictly decreasing supply functions. This yields a potential dilemma for any regulation in favour of the non-strategic buyer. On the one hand, a sufficiently powerful regulator could effectively prevent collusion if he restricts the set of admissible supply functions to nondecreasing supply functions only. On the other hand, the competitive Nash equilibrium—which is optimal from the buyer's perspective—also requires the admissibility of decreasing supply functions. To be specific, the decreasing supply functions required for the competitive Nash equilibrium must have a significantly steeper negative slope than the decreasing supply functions required for the collusive Nash equilibrium. This finding is in stark contrast to the zero-cost model for which the competitive outcome can only be approximated through strictly increasing supply functions whose slope approaches plus infinity.

The remainder of this paper is structured as follows. Section 2 discusses related literature. Section 3 constructs strategic games of supply function competition with quadratic or zero costs. I derive Theorem 1 in Section 4 which characterizes all symmetric Nash equilibria. To pin down the set of all possible market-clearing prices that can be generated by some symmetric Nash equilibrium, Section 5 analyses the Möbius structure of the extreme-point price function. Section 6 derives the unique competitive Nash equilibrium and Section 7 derives the unique collusive Nash equilibrium. Section 8 concludes. The Mathematical Appendix contains additional details for the proof of Theorem 1.

2 Relationship to the literature

Robson (1981) and Turnbull (1983). Most closely related to the class of games considered in this paper are the static games of affine supply function competition in (i) Robson (1981) for zero-costs and N sellers and (ii) in Turnbull (1983) for quadratic costs and two sellers. In Zimper (2026) I had already revisited these two pioneering contributions to resolve the frictions between Turnbull's (1983) results and the corresponding analysis in the seminal article by Klemperer and Meyer (1989). Although Zimper (2026) is mainly concerned with the correction of Turnbull's (1983) erroneous characterization of the unique Nash equilibrium which obtains under demand uncertainty, a side result also corrects (and extends to N sellers) Turnbull's (1983) erroneous characterization of the multitude of symmetric Nash equilibria which obtains under demand certainty. That said, the analysis in Zimper (2026) of symmetric Nash equilibria under demand certainty was restricted to nondecreasing supply functions only whereby the formal argument was, basically, only a first-order condition (=FOC) analysis ignoring, resp. taking for granted, the second-order condition (=SOC) for nondecreasing supply functions. As a generalization of Proposition 1 in Zimper (2026), this paper's Theorem 1 is based on a comprehensive SOC analysis which fully characterizes all symmetric Nash equilibria for all decreasing and nondecreasing affine supply functions.

The "anything goes" arguments in Klemperer and Meyer (1989). In their seminal article Klemperer and Meyer (1989) argue—as point of departure—that "anything goes" in models of supply function competition as long as the non-strategic buyer's demand is deterministic. More precisely, Klemperer and Meyer (1989) offer two different "anything goes" arguments. Their main argument in Chapter 2 and Footnote 10 constructs symmetric Nash equilibria for a two-seller model in which a seller's supply function would at first decrease—running parallel to a halved affine demand function—and later increase at a constant slope (satisfying a local FOC condition) to cross the halved affine demand function at the desired outcome. This FOC argument requires a change in slopes and only establishes equilibrium outcomes at which marginal costs are less than the equilibrium price. A second "anything goes" argument is more general as it shows how any prices that result in nonnegative profits can be supported by some Nash equilibrium in discontinuous supply functions (cf. Footnote 11 in Klemperer and Meyer 1989).

Both "anything goes" arguments of Klemperer and Meyer (1989) are not applicable to our model because affine supply functions are continuous and have a constant slope. The really interesting finding of our "almost anything goes" analysis is that affine supply functions with their innocuous structure—being continuous and having a constant slope—can nevertheless generate through a symmetric Nash equilibrium almost every price that yields a nonnegative profit.

The "demand uncertainty implies uniqueness" argument in Klemperer and Meyer (1989). As their celebrated main result Klemperer and Meyer (1989) argue that sufficiently rich demand uncertainty might drastically reduce the multitude of Nash equilibria; under suitable conditions even to a unique Nash equilibrium in linear supply functions. Klemperer and Meyer's (1989) characterization of unique linear Nash equilibria under demand uncertainty (or over time) has been used in models of electricity markets (e.g., Green and Newbery 1992; Bolle 1992; Holmberg and Newbery 2010; Holmberg, Newbery, and Ralph 2013) and also appears in the model for the auction of treasury notes by Back and Zender (1993). Because of this uniqueness result under demand uncertainty, several authors restrict attention to games with linear supply functions only (cf., e.g., Escrihuela-Villar, Gutiérrez-Hita, and Vicente-Pérez 2020 and references therein).

Formally, the uniqueness analysis of Klemperer and Meyer (1989) only applies to (smooth) increasing supply functions that go through the origin thereby excluding arbitrary affine supply functions. However, Proposition 2 in Zimper (2026) shows that there also exists a unique Nash equilibrium under demand uncertainty for this paper's model of affine supply function competition which is linear, i.e., goes through the origin.[1] One could argue that demand uncertainty is the more realistic assumption than demand certainty. But even accepting the prevalence of demand uncertainty, the question is whether the

[1]For his duopoly economy with quadratic costs, Turnbull (1983) did already argue that demand uncertainty implies uniqueness of the equilibrium. However, due to a mathematical error, Turnbull's (1983) characterization of this unique equilibrium supply function was not linear, i.e., did not go through the origin. Proposition 2 in Zimper (2026) corrects Turnbull's mathematical error and extends his analysis to the general case of N sellers.

sellers choose in reality their supply functions before or after they learn the true demand. This point is made by Menezes and Quiggin (2025) who distinguish between markets with *ex ante* versus *ex post* decision making whereby they argue that both kinds of markets are relevant in real life.[2] In the terminology of these authors, the present paper's analysis applies to markets with *ex post* decision making.

Collusion supported by subgame-perfect Nash equilibria in supergames. The bulk of monopolistic collusion models in the literature are infinitely repeated stage games, i.e., *supergames* in the terminology of Friedman (1971). Based on folk-theorems for the existence of *subgame-perfect Nash equilibria* (=SPNE) in supergames, collusive SPNE have been described for infinitely repeated games of (i) Bertrand price competition (e.g., Chapter 6.3 in Tirole 1988; Rotemberg and Saloner 1986; Athey, Bagwell, and Sanchirico 2004), (ii) Cournot output competition (e.g., Green and Porter 1984), and (iii) linear supply function competition a la Klemperer and Meyer (1989) (e.g., Ciarreta and Gutiérrez-Hita 2012; Celen and Saglam 2022). All these models rely on punishment strategies (e.g., *trigger* strategies) which only become credible threats—thereby supporting collusive SPNE—for infinite but not for finite horizon models.[3] The analysis in this paper complements this literature on tacit collusion by showing that—seemingly innocuous—affine supply functions are able to generate monopoly prices in a static game without any obvious threat to punish deviating behaviour as long as strictly decreasing supply functions are admissible.

[2]Menezes and Quiggin (2025, p.81) write: "In other markets, such as the Australian National Electricity Market, however, bids are submitted up to 5 min prior to dispatch, so that bidders are aware of demand."

[3]For an alternative model of infinite horizon (continuous time) supply function competition, see the differential game of Menezes and Pereira (2017). By construction, stationary Markovian feedback equilibria for differential games would not trigger any punishment for deviating behavior and does therefore not (as far as I can see) generate any collusion beyond collusion in static games.

3 Strategic games of affine supply function competition

3.1 Market demand, strategies

The non-strategic buyer has a strictly decreasing demand function

$$D(p) = z - mp$$

for fixed parameters $z, m > 0$.

A strategic seller $i \in \{1, ..., N\}$, $N \geq 2$, has to choose some *affine* supply function

$$S_i(p) = -a_i + \beta_i p.$$

Because I allow for arbitrary real values of the intercept $-a_i$ and slope β_i, the set of individual strategies of i is given as

$$\Sigma_i = \mathbb{R} \times \mathbb{R}$$

with generic element $\sigma_i = (a_i, \beta_i)$. By choosing $\sigma_i = (a_i, \beta_i)$, seller i offers to sell the amount $S_i[\sigma_i](p^*) = -a_i + \beta_i p^*$ at per-unit price p^* whenever the Walrasian auctioneer declares p^* as market price.

From the perspective of seller i only the aggregate parameter choices of his opponents affect his profit. I therefore generically associate a strategy profile $(\sigma_i, \sigma_{-i}) \in \Sigma = \times_{j=1}^{N} \Sigma_j$ with

$$(a_i, A_{-i}, \beta_i, B_{-i})$$

such that

$$A_{-i} = \sum_{j \neq i} a_j \text{ and } B_{-i} = \sum_{j \neq i} \beta_j.$$

3.2 Market price

The price $p^* \in \mathbb{R}$ clears the market under strategy profile

$$\sigma = (a_i, A_{-i}, \beta_i, B_{-i})$$

iff the aggregate supply of all sellers meets the market demand, i.e., iff

$$\sum_{i=1}^{N} S_i [\sigma_i] (p^*) = D (p^*) \Leftrightarrow - (a_i + A_{-i}) + (\beta_i + B_{-i}) p^* = z - mp^*.$$

There exists a unique well-defined market-clearing price given as

$$p^* = \frac{z + a_i + A_{-i}}{m + \beta_i + B_{-i}}$$

iff $m + \beta_i + B_{-i} \neq 0$. In this case the Walrasian auctioneer announces p^* as market price.

Whenever $m + \beta_i + B_{-i} = 0$ (i.e., whenever the slope of the demand function coincides with the slope of the aggregate supply function) two things may happen. Firstly, the demand function and the aggregate supply function coincide so that any $p^* \in \mathbb{R}$ is market-clearing. This is the case iff $-a_i - A_{-i} = z$. Secondly, if $-a_i - A_{-i} \neq z$ the demand function and the aggregate supply function just run parallel without ever intersecting. Then there does not exist any market-clearing price at all. For both cases I will simply assume that the Walrasian auctioneer will announce a market price of zero.

To summarize, the market price under any strategy-profile $\sigma = (a_i, A_{-i}, \beta_i, B_{-i})$ is defined as

$$p [\sigma] = \begin{cases} \frac{z + a_i + A_{-i}}{m + \beta_i + B_{-i}} & \text{if } m + \beta_i + B_{-i} \neq 0, \\ 0 & \text{else.} \end{cases}$$

3.3 Profit and strategic game

All sellers face quadratic (or zero) costs $C (q_i) = \frac{c}{2} (q_i)^2$ in their supplied amount $q_i = S_i [\sigma_i] (p [\sigma])$ for some fixed cost-parameter $c \geq 0$. Seller i's profit under strategy-profile $(\sigma_i, \sigma_{-i}) \in \Sigma$ is formally given as

$$\pi_i [\sigma_i, \sigma_{-i}] = \begin{cases} p [\sigma] S_i [\sigma_i] (p [\sigma]) - \frac{c}{2} (S_i [\sigma_i] (p [\sigma]))^2 & \text{if } m + \beta_i + B_{-i} \neq 0, \\ -\frac{c}{2} (S_i [\sigma_i] (0))^2 & \text{else.} \end{cases} \tag{1}$$

For any fixed $(\sigma_i, \sigma_{-i}) = (a_i, A_{-i}, \beta_i, B_{-i})$ with $m + \beta_i + B_{-i} \neq 0$ the profit (1) becomes

$$\begin{aligned} \pi_i &= \left(\frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m} \right) \left(-a_i + \beta_i \frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m} \right) \\ &\quad - \frac{c}{2} \left(-a_i + \beta_i \frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m} \right)^2 . \end{aligned}$$

For fixed (a_i, A_{-i}, B_{-i}) the market-price—and therefore the profit—is (typically) discontinuous in $\beta_i \in \mathbb{R}$ at $\beta_i = -m - B_{-i}$ with negative value

$$\pi_i = -\frac{c}{2}(a_i)^2 \leq 0.$$

The economy is defined as the strategic game

$$\Gamma = \langle \Sigma_i, U_i \rangle_{i \in \{1,...,N\}}$$

such that seller i's utility $U_i(\sigma_i, \sigma_{-i})$, $(\sigma_i, \sigma_{-i}) \in \Sigma$, is given as his profit $\pi_i[\sigma_i, \sigma_{-i}]$. The strategy profile $\sigma^* \in \Sigma$ is a Nash equilibrium of Γ iff, for all $i \in \{1, ..., N\}$,

$$U_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq U_i\left(\sigma_i, \sigma_{-i}^*\right) \text{ for all } \sigma_i \in \Sigma_i.$$

4 Symmetric Nash equilibria

4.1 Main result

Theorem 1 characterizes all symmetric Nash equilibria of a given game Γ in dependence on the values of the slope, denoted β^*, of the sellers' common equilibrium supply function. The main analytical challenge is to determine the subset of the real line, denoted $\mathcal{B}$, which contains all the equilibrium values β^*.

Theorem 1. *The symmetric Nash equilibria $\sigma^* = (a^*, \beta^*)_{i=1,..,N}$ of Γ are given as*

$$a^* = \frac{z\left[(cm - (N-2))\beta^* - m + (N-1)c(\beta^*)^2\right]}{m + ((N-1)\beta^* + m)(N + cm)} \tag{2}$$

for any

$$\beta^* \in \mathcal{B} = \left(-\infty, -\frac{1}{N-1}\left(m + \frac{2}{c}\right)\right] \cup \left[-\frac{m}{N-1}, -\frac{m}{N}\right) \cup \left(-\frac{m}{N}, \infty\right).$$

To simplify the subsequent analysis, let me introduce a more compact notation for the set $\mathcal{B}$.

Notational conventions. *Fix the two points*

$$\begin{aligned}\beta^1 &= -\frac{1}{N-1}\left(m+\frac{2}{c}\right),\\ \beta^2 &= -\frac{m}{N-1}\end{aligned}$$

and rewrite $\mathcal{B}$ as the union

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$$

such that

$$\begin{aligned}\mathcal{B}_1 &= \begin{cases} \left(-\infty, \beta^1\right] & \text{if } c>0 \\ \emptyset & \text{if } c=0 \end{cases},\\ \mathcal{B}_2 &= \left[\beta^2, \infty\right) \setminus \left\{-\frac{m}{N}\right\}.\end{aligned}$$

Sketch of the Proof of Theorem 1 (with additional details relegated to the Appendix).[4] **The FOC part.** Any best response (a_i^*, β_i^*) against fixed (A_{-i}, B_{-i}) has to satisfy in a symmetric Nash equilibrium the two FOCs

$$\frac{\partial \pi_i}{\partial a_i} = 0, \tag{3}$$

$$\frac{\partial \pi_i}{\partial \beta_i} = 0. \tag{4}$$

Solving the first FOC for a_i^* allows us to express a_i^* as a differentiable function $a_i^*(\beta_i)$ in β_i. By the envelope theorem, we obtain for the reaction of i's 'value' function $\pi_i\left(a_i^*(\beta_i), \beta_i\right)$ to a change in parameter β_i

$$\frac{d\pi_i}{d\beta_i}\left(a_i^*(\beta_i), \beta_i\right) = \frac{\partial \pi_i}{\partial a_i}\frac{da_i^*(\beta_i)}{d\beta_i} + \frac{\partial \pi_i}{\partial \beta_i} = \frac{\partial \pi_i}{\partial \beta_i}\left(a_i^*(\beta_i), \beta_i\right) \tag{5}$$

because of $\frac{\partial \pi_i}{\partial a_i} = 0$ at $a_i^*(\beta_i)$.

[4] Theorem 1 generalizes a previous result in Zimper (2026, Proposition 1), which was restricted to nondecreasing affine supply functions only. In particular, the analysis of the SOC part is completely novel as it comes with the full characterization of all symmetric Nash equilibria for all—decreasing as well as nondecreasing—affine supply functions.

Importantly, the model comes with the specific structure that the first FOC (3) implies the second FOC (4) (an insight due to Turnbull 1983). That is, $\frac{\partial \pi_i}{\partial a_i}(a_i^*, \beta_i^*) = 0$ implies $\frac{\partial \pi_i}{\partial \beta_i}(a_i^*, \beta_i^*) = 0$ to the effect that (5) reduces to

$$\frac{d\pi_i}{d\beta_i}(a_i^*(\beta_i), \beta_i) = 0. \tag{6}$$

Consequently, the value function $\pi_i(a_i^*(\beta_i), \beta_i)$ remains constant in all values β_i for which $\frac{d\pi_i}{d\beta_i}(a_i^*(\beta_i), \beta_i)$ is well-defined. Let's collect these β_i in the domain $\mathcal{D}$. Note that $\mathcal{D}$ excludes $\beta_i = -m - B_{-i}$ at which π_i is discontinuous. Additionally, $\mathcal{D}$ excludes any values β_i which give rise to zero denominators in $\frac{d\pi_i}{d\beta_i}(a_i^*(\beta_i), \beta_i)$.

To summarize so far: i's utility against fixed (A_{-i}, B_{-i}) remains constant for all parameter values $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, whereby the two FOCs (3) and (4)—as a necessary condition for a maximum in a symmetric Nash equilibrium—are satisfied at all these parameter values.

Having characterized $a_i^*(\beta_i)$ through the first FOC (3) yields for every symmetric strategy-profile $(a^*, \beta^*)_{i=1,..,N}$ the desired characterization of

$$a^* = \frac{z\left[(cm - (N-2))\beta^* - m + (N-1)c(\beta^*)^2\right]}{m + ((N-1)\beta^* + m)(N + cm)}$$

in dependence on β^*. However, although we know that both FOCs are satisfied for all (a^*, β^*), $\beta^* \in \mathcal{D}$, we do not know yet whether these FOCs correspond to utility maxima or -minima (or to some saddle point). This brings us to the SOC analysis.

The SOC part. Because any maximal utility remains constant on the segment $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, the standard sufficiency condition for a (local) maximum in two variables (a_i^*, β_i), which would exclude a saddle-point, is not applicable.[5] However, the specific geometric structure of the problem—according to which $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, correspond to the constant top of a 'ridge' or to the constant bottom of a 'valley' (or

[5] That is, we have

$$\frac{\partial^2 \pi_i}{\partial a_i \partial a_i}\frac{\partial^2 \pi_i}{\partial \beta_i \partial \beta_i} = \left(\frac{\partial^2 \pi_i}{\partial a_i \partial \beta_i}\right)^2$$

evaluated at $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, so that the sufficiency condition for ruling out a saddle-point is never satisfied.

both, i.e., a flat plane)—implies that the SOC for a global maximum at $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, holds iff

$$\frac{\partial^2 \pi_i}{\partial a_i \partial a_i}(a_i^*(\beta_i), \beta_i) < 0.$$

Conversely, the SOC for a global minimum at $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, holds iff

$$\frac{\partial^2 \pi_i}{\partial a_i \partial a_i}(a_i^*(\beta_i), \beta_i) > 0.$$

The second-order derivative in a_i comes with the simple structure

$$\frac{\partial^2 \pi_i}{\partial a_i \partial a_i} = (-)\left(2(B_{-i}+m) + c(B_{-i}+m)^2\right)\left(\frac{1}{(\beta_i + B_{-i}+m)^2}\right)$$

so that

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial a_i \partial a_i}(a_i^*(\beta_i), \beta_i) &< (>)\, 0 \Leftrightarrow \\ 2(B_{-i}+m) + c(B_{-i}+m)^2 &> (<)\, 0. \end{aligned}$$

In other words, whether the SOC for a maximum or a minimum holds at $(a_i^*(\beta_i), \beta_i)$, $\beta_i \in \mathcal{D}$, exclusively depends on the values of the fixed parameters B_{-i}, m, c. For a symmetric Nash equilibrium with $B_{-i} = (N-1)\beta^*$, I show in the Appendix that the SOC holds for a maximum at $(a_i^*(\beta^*), \beta^*)$ iff

$$\beta^* \in \left(-\infty, \beta^1\right) \cup \left(\beta^2, \infty\right) \setminus \left\{-\frac{m}{N}\right\}. \tag{7}$$

Conversely, any symmetric strategy-profile with $B_{-i} = (N-1)\hat{\beta}$ yields the SOC for a minimum at $\left(a_i^*\left(\hat{\beta}\right), \hat{\beta}\right)$ iff

$$\hat{\beta} \in \left(\beta^1, \beta^2\right) \cap \mathcal{D}. \tag{8}$$

So, whereas the symmetric strategy-profiles $(a_i^*(\beta^*), \beta^*)$ with β^* belonging to (7) are mutually best responses, the symmetric strategy-profiles $\left(a_i^*\left(\hat{\beta}\right), \hat{\beta}\right)$ with $\hat{\beta}$ belonging to (8), are mutually worst responses.

This only leaves the question what happens at the symmetric strategy-profiles $(a_i^*(\beta), \beta)$ with

$$\beta = \beta^1 \text{ and } \beta = \beta^2.$$

The answer is straightforward: these symmetric strategy-profiles are both mutually best and worst responses as any seller i's utility is pushed to zero by his opponents' choices irrespective of his own supply function choice.

This is easy to see for $\beta = \beta^1$. If $\beta_j^* = \beta^1 = -\frac{m}{N-1}$ for $j \neq i$, we have

$$\begin{aligned} a_j^* &= \frac{z\left[(cm-(N-2))\,\beta_j^* - m + (N-1)\,c\left(\beta_j^*\right)^2\right]}{m + \left((N-1)\,\beta_j^* + m\right)(N+cm)} \\ &= -\frac{z}{N-1}, \end{aligned}$$

which yields $(N-1)\left(-a_j^*\right) = z$. Consequently, the aggregate supply of i's opponents

$$(N-1)\left(-a_j^* + \beta_j^* p\right) = z - mp$$

coincides with the buyer's demand curve

$$D(p) = z - mp$$

so that i's residual demand function becomes constant zero.

Let now $c > 0$ and consider the case $\beta = \beta^2$. If $\beta_j^* = \beta^2 = -\frac{1}{N-1}\left(m + \frac{2}{c}\right)$ for $j \neq i$, we obtain

$$B_{-i} = -\left(m + \frac{2}{c}\right) \tag{9}$$

as well as

$$\begin{aligned} a_j^* &= \frac{z\left[(cm-(N-2))\,\beta_j^* - m + (N-1)\,c\left(\beta_j^*\right)^2\right]}{m + \left((N-1)\,\beta_j^* + m\right)(N+cm)} \\ &= -\frac{z}{N-1} \end{aligned}$$

so that

$$A_{-i} + z = 0. \tag{10}$$

Substituting (9) and (10) in i's profit function, yields a zero utility

$$\begin{aligned} \pi_i &= \left(\frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m}\right)\left(-a_i + \beta_i \frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m}\right) - \frac{c}{2}\left(-a_i + \beta_i \frac{a_i + A_{-i} + z}{\beta_i + B_{-i} + m}\right)^2 \\ &= 0 \end{aligned} \tag{11}$$

irrespective of i's choice of (a_i, β_i) provided that $\beta_i + B_{-i} + m \neq 0$ and $c > 0$.□□

4.2 Selected special cases and modified games

The following corollary to Theorem 1 restates Robson's (1981) characterization of symmetric Nash equilibria in affine supply functions for his zero-cost model. The only difference is that Robson did not include the value $\beta^* = -\frac{m}{N-1}$ at which all sellers are indifferent between all their supply function choices.

Corollary 1. *For $c = 0$ the symmetric Nash equilibria $\sigma^* = (a^*, \beta^*)_{i=1,..,N}$ of Γ become*

$$a^* = \frac{-z\left[(N-2)\,\beta^* + m\right]}{(N+1)\,m + N\,(N-1)\,\beta^*}$$

for any

$$\beta^* \in \mathcal{B} = \left[-\frac{m}{N-1}, -\frac{m}{N}\right) \cup \left(-\frac{m}{N}, \infty\right).$$

The large multitude of Nash equilibria from Theorem 1 is drastically reduced if we consider modified games which restrict the set of admissible supply functions by fixing exogenously selected parameter values.

Corollary 2. *Consider the modified game $\Gamma^{\beta=0}$ such that only constant affine supply functions, i.e., $\beta_i = \beta = 0$ for all i, are admissible. Then $\beta^* = 0$ in Theorem 1 implies the unique symmetric Nash equilibrium $\sigma^* = (a^*)_{i=1,..,N}$ with*

$$a^* = \frac{-z}{(N+1)+cm},$$

which corresponds to the Nash equilibrium of Cournot output competition with quadratic (or zero) costs.

More generally than setting $\beta = 0$, Menezes and Quiggin (2012; 2020) fix a nonnegative slope $\beta \in [0, \infty)$ which they interpret as an exogenously given measure for the industry's level of competitiveness between the poles of Cournot, i.e., $\beta = 0$, and Bertrand, i.e.,

$\beta \to \infty$, competition. The reason for the latter interpretation is that for the zero-cost model considered by these authors the equilibrium prices converge to zero if $\beta \to \infty$ as in a model of Betrand price competition with constant *inverse* demand functions. We will see that for non-zero quadratic costs this two-pole interpretation is no longer available as the smallest possible equilibrium price is no longer approximated for $\beta \to \infty$ but obtains for a strictly, rather steep, negative slope.[6] Irrespective of the ensuing more complex interpretation of $\beta \in \mathcal{B}$ as a fixed competitiveness measure, the following corollary adopts and extends the approach of Menezes and Quiggin (2012; 2020) to arbitrary exogenously fixed values of the slope.

Corollary 3. *Consider the modified game Γ^{β} such that only affine supply functions with fixed slope $\beta_i = \beta$ for all i are admissible. Then $\beta^* = \beta$ in Theorem 1 implies that there exists a symmetric Nash equilibrium iff $\beta \in \mathcal{B}$. Moreover, any such Nash equilibrium $\sigma^* = (a^*)_{i=1,..,N}$ is uniquely pinned down as*

$$a^* = \frac{z\left[(cm-(N-2))\,\beta - m + (N-1)\,c\,(\beta)^2\right]}{m+((N-1)\,\beta+m)\,(N+cm)}.$$

Next suppose that only nondecreasing linear supply functions are admissible, i.e., $a = a_i = 0$ and $\beta_i \geq 0$ for all i. Then all Nash equilibria correspond to any positive β^* which is a solution to the following quadratic equation

$$\begin{aligned} 0 &= \frac{z\left[(cm-(N-2))\,\beta^* - m + (N-1)\,c\,(\beta^*)^2\right]}{m+((N-1)\,\beta^*+m)\,(N+cm)} \Rightarrow \\ 0 &= (\beta^*)^2 + \frac{cm-(N-2)}{(N-1)\,c}\beta^* - \frac{m}{(N-1)\,c}, \end{aligned}$$

whereby $\beta^* > 0$ implies $\beta^* \in \mathcal{B}$.

Corollary 4. *Consider the modified game $\Gamma^{a=0}$ such that only nondecreasing linear supply functions are admissible. For $c > 0$, $a^* = 0$ in Theorem 1 implies that there*

[6] I discuss the details in Section 6 "Competitive Nash equilibria".

exists a unique symmetric Nash equilibrium $\sigma^* = (\beta^*)_{i=1,..,N}$ *with*

$$\beta^* = \frac{(N-2-cm)}{2c(N-1)} + \sqrt{\left(\frac{(N-2-cm)}{2c(N-1)}\right)^2 + \frac{m}{c(N-1)}}\,. \tag{12}$$

Remark 1. Demand uncertainty and expected profit-maximization. The fact that the unique β^* in (12) is the same for any $z, z' > 0$ drives the result that the multitude of Nash equilibria of Theorem 1 in (nondecreasing) affine supply functions reduces to a unique Nash equilibrium in a linear supply function—with β^* given by (12)—(i) if the intercept of the demand function is no longer certain but becomes random and if (ii) the sellers are expected profit maximizers. The intuition for this uniqueness result is straightforward: If there exists the same best response with $a^* = 0$ and β^* against both z and z', this best response will also be the best response of an expected profit maximizing seller who has to put positive weight on z and z'. For the details of this result—and it's relationship to the uniqueness analysis in Klemperer and Meyer (1989)—, see Proposition 2 in Zimper (2026).

5 Almost anything goes

This section identifies all market-clearing prices that can be generated by the symmetric Nash equilibria of any given game Γ. Before I proceed with formal derivations, two qualitative observations are in order. Firstly, the resulting market price will never be negative in a symmetric Nash equilibrium. Negative prices mean that the sellers would have to pay the buyer to purchase their product. But this would be strictly worse than choosing a zero supply function. Secondly, sellers will always sell a nonnegative amount (i.e., they will never buy the good in a symmetric Nash equilibrium). Else, their profit would become negative which could, again, be avoided by choosing a zero supply function.

5.1 The "anything goes" benchmark case

Suppose, for the moment being, that Γ is modified to Γ' such that not only affine supply functions are admissible but also supply functions that are affine almost everywhere (i.e., that only allow for discontinuities on a subset of prices with Lebesgue measure zero). Consider the symmetric strategy profile

$$\sigma^{p^*} = \left(S^{p^*}(p)\right)_{i=1,\ldots,N}$$

such that the supply function, being discontinuous at exactly one point, is given as

$$S^{p^*}(p) = \begin{cases} \frac{1}{N-1}(z - mp) & \text{if } p \neq p^*, \\ \frac{z-mp^*}{N} & \text{if } p = p^*. \end{cases}$$

Then the strategy-profile σ^{p^*} constitutes a symmetric Nash equilibrium of Γ' iff the price p^* satisfies (i) nonnegative supply

$$\frac{z - mp^*}{N} \geq 0 \Leftrightarrow p^* \leq \frac{z}{m},$$

as well as (ii) nonnegative profit at positive supply

$$\begin{aligned} \pi_i\left[\sigma^{p^*}\right] &= p^*\left(\frac{z-mp^*}{N}\right) - \frac{c}{2}\left(\frac{z-mp^*}{N}\right)^2 \geq 0 \Rightarrow \\ p^* &\geq \frac{zc}{2N+cm}. \end{aligned}$$

The argument is simple. If i's opponents choose $S^{p^*}(p)$, then their aggregate supply function coincides with the buyer's supply function at any price $p \neq p^*$ because of

$$(N-1)\frac{1}{N-1}(z - mp) = z - mp.$$

As a result, i faces a constant zero residual demand—and therefore a zero-profit—at any price $p \neq p^*$. Compared to this zero-profit, he can secure himself a market-clearing price p^* and supplied amount of $\frac{z-mp^*}{N} \geq 0$ if he also chooses $S^{p^*}(p)$. He has a strict (weak) incentive to do so whenever p^* and $\frac{z-mp^*}{N} \geq 0$ yield a positive (nonnegative) profit.

I will henceforth refer to the set of market-clearing equilibrium prices of Γ'

$$\mathcal{P} = \left[\frac{zc}{2N+cm}, \frac{z}{m}\right] \tag{13}$$

as the "anything goes" benchmark case.

5.2 The Möbius transformation

Coming back to our original game Γ—with affine supply functions that are continuous everywhere—, the above construction of the "anything goes" result through discontinuous supply functions is not available. Also any FOC-based argument of Chapter 2/Footnote 10 in Klemperer and Meyer (1989), which would require kinks or at least different slopes for admissible supply functions, is not available. This raises the following question:

> Which market-clearing prices can be supported by the symmetric Nash equilibria from Theorem 1?

To answer this question, I am going to derive two observations which I will use for the proof of Proposition 1. As point of departure, observe that a symmetric Nash equilibrium $\sigma^* = (a^*, \beta^*)_{i=1,..,N}$ from Theorem 1 generates the market-clearing price

$$p[\sigma^*] = \frac{z + Na^*}{m + N\beta^*} = \frac{z + N\left(\frac{z\left[(cm-(N-2))\beta^* - m + (N-1)c(\beta^*)^2\right]}{(N+1)m + N(N-1)\beta^* + (N-1)\beta^* cm + cm^2}\right)}{m + N\beta^*} \tag{14}$$

where $\beta^* \in \mathcal{B}$ ensures that $m + N\beta^* \neq 0$.

Observation 1. *The equilibrium price (14) simplifies to*

$$p[\sigma^*] = \frac{z\left[(N-1)c\beta^* + 1 + cm\right]}{(N-1)(N+cm)\beta^* + m(N+1+cm)}. \tag{15}$$

Proof of Observation 1. Rewrite the inner denominator in (14) as

$$D(\beta^*) = (N+1)m + N(N-1)\beta^* + (N-1)\beta^* cm + cm^2.$$

A direct expansion shows

$$D(\beta^*) + N\left[(cm - (N-2))\beta^* - m + (N-1)c(\beta^*)^2\right] = (N\beta^* + m)\left[(N-1)c\beta^* + 1 + cm\right].$$

as both sides become $N(N-1)c(\beta^*)^2 + N\beta^* + cm(2N-1)\beta^* + m + cm^2$. Consequently,

$$p(\beta^*) = \frac{z\left(\frac{(N\beta^* + m)[(N-1)c\beta^* + 1 + cm]}{D(\beta^*)}\right)}{m + N\beta^*},$$

which yields the desired result after cancelling $m+N\beta^{*}\neq 0$.□□

Define now

$$p(\beta)=\frac{z\left[(N-1)\,c\beta+1+cm\right]}{(N-1)\,(N+cm)\,\beta+m\,(N+1+cm)} \tag{16}$$

for all $\beta\in\mathcal{D}$ (i.e., for all $\beta\in\mathbb{R}$ for which $p(\beta)$'s denominator does not vanish). The important fact to observe is that (16) has the mathematical structure of a so-called *Möbius transformation*[7]

$$p(\beta)=\frac{A\beta+B}{C\beta+D}$$

with

$$\begin{aligned} A &= z\,(N-1)\,c, \\ B &= z\,(1+cm)\,, \\ C &= (N-1)\,(N+cm)\,, \\ D &= m\,(N+1+cm) \end{aligned}$$

whereby

$$AD-BC=-zN\,(N-1)<0$$

implies the condition

$$AD-BC\neq 0$$

for a non-constant function $p(\beta)\neq\frac{A}{C}$.

Next, denote by $\mathbb{R}^{*}=\mathbb{R}\cup\{\infty\}$ the *Alexandroff one-point compactification of the real line*. This topological space can, e.g., be generated by the *Chordal metric*: for all $x,y\in\mathbb{R}\cup\{\infty\}$

$$d(x,y)=\begin{cases}\frac{|x-y|}{\sqrt{(1+x^{2})}\sqrt{1+y^{2}}} & \text{if } x,y\in\mathbb{R},\\ \frac{1}{\sqrt{(1+x^{2})}} & \text{if } x\in\mathbb{R},\ y=\infty\\ 0 & \text{if } x=y=\infty.\end{cases}$$

The basic idea is here to *stereographically project* every point $x\in\mathbb{R}\cup\{\infty\}$ to a unique point $(u_x,v_x)\in\mathbb{R}^{2}$ on a circle such that the point 0 corresponds to the circle's south-pole

[7] For very deep details (way beyond the topic of the present paper) on Möbius transformations, see, e.g., the Handbook of Complex Analysis (Kühnau (Ed.) 2002).

$(0,0)$ and ∞ corresponds to the north-pole (e.g., $(0,1)$ for the circle of diameter one). The Chordal distance between x, y is then defined as the Euclidean distance between the two points (u_x, v_x) and (u_y, v_y) in $\mathbb{R}^2$.[8]

For this topological space we can now define the continuous function $p : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$p(\beta) = \begin{cases} p_{asym} & \text{if } \beta = \infty, \\ \infty & \text{if } \beta = -\frac{D}{C}, \\ \frac{z[(N-1)c\beta+1+cm]}{(N-1)(N+cm)\beta+m(N+1+cm)} & \text{else}, \end{cases} \tag{17}$$

with

$$\begin{aligned} p_{asym} &= \frac{z\left[(N-1)c + \frac{1+cm}{\infty}\right]}{(N-1)(N+cm) + \frac{m(N+1+cm)}{\infty}} = \frac{zc}{N+cm}, \\ -\frac{D}{C} &= -\frac{m(N+1+cm)}{(N-1)(N+cm)}. \end{aligned}$$

I refer to the Möbius transformation (17) as the *'extreme-point' price function* because it corresponds, for all $\beta \in \mathcal{D}$, to the symmetric strategy-profiles $\sigma = (a,\beta)_{i=1,..,N}$ which satisfy the FOC for a maximum or a minimum (or both). The next collection of results applies well-known properties of a Möbius transformation to our set-up; (I omit any formal proofs as these results are not difficult to verify).

Observation 2.

(i) *The Möbius transformation $p(\beta)$ has the unique pole*

$$p\left(\beta_{pole}\right) = \infty \Rightarrow \beta_{pole} = -\frac{m(N+1+cm)}{(N-1)(N+cm)}$$

which satisfies

$$\beta^1 < \beta_{pole} < \beta^2.$$

[8]Although the topology bible for Economists, Aliprantis and Border (2006), mentions that the *Alexandroff one-point compactification* results in a homeomorphism between the extended real line and a circle, for more details the authors only recommend: "Look up stereographic projection in a good dictionary" (p.57). I guess in our brave new world one must recommend: "Just Google 'How can the Chordal metric for the one-point extended real line be constructed?'", and their AI will provide the answer.

(ii) *The first derivative of* $p(\beta)$ *becomes*

$$\frac{dp}{d\beta} = \frac{-zN(N-1)}{[(N-1)(N+cm)\beta + m(N+1+cm)]^2} < 0$$

(wherever defined) so that $p(\beta)$ *is strictly decreasing on both sides of it's pole.*

(iii) *The Möbius transformation* $p : \mathbb{R}^* \to \mathbb{R}^*$ *is one-to-one and onto (i.e., bijective).*

(iv) *By (ii) and (iii), we must have for all* $\beta', \beta'' \in \mathbb{R}$ *with*

$$\beta' < \beta_{pole} < \beta''$$

that

$$p(\beta') < p_{asym} < p(\beta'') .$$

Example 1. Let $N = 2$, $z = 10$, $m = 1$, $c = 1$ to obtain the Möbius transformation

$$p(\beta) = \frac{10\beta + 20}{3\beta + 4}$$

with unique pole

$$\beta_{pole} = -\frac{4}{3}$$

and

$$p_{asym} = \frac{10}{3}.$$

Figure 1 plots this function.□

5.3 Almost "anything goes"

Based on the properties of the Möbius transformation (17), the following 'almost anything goes' result is easy to obtain.

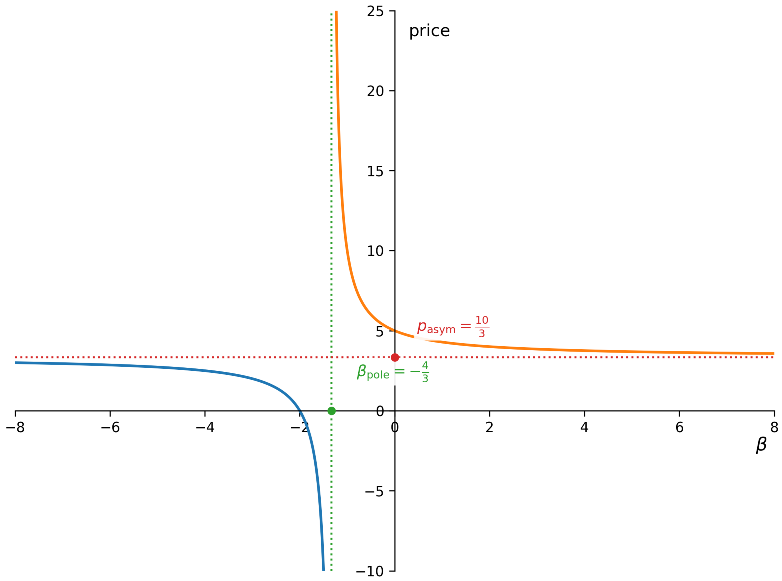

Figure 1: The Möbius transformation $p(\beta) = \frac{10\beta+20}{3\beta+4}$ with $\beta_{pole} = -\frac{4}{3}$ and $p_{asym} = \frac{10}{3}$

Proposition 1. *Let $c > 0$. The symmetric Nash equilibria $\sigma^* = (a^*, \beta^*)_{i=1,..,N}$ from Theorem 1 can generate any market-clearing price*

$$\begin{aligned} p[\sigma^*] &\in \left[p\left(\beta^1\right), p\left(\beta^2\right)\right] \setminus \left\{p_{asym}, p\left(-\frac{m}{N}\right)\right\} \\ &= \left[\frac{zc}{2N+cm}, \frac{z}{m}\right] \setminus \left\{\frac{zc}{N+cm}, \frac{z(N+cm)}{m(2N+cm)}\right\} \\ &= \mathcal{P} \setminus \left\{\frac{zc}{N+cm}, \frac{z(N+cm)}{m(2N+cm)}\right\}. \end{aligned}$$

For $c = 0$ this set of possible equilibrium prices reduces to

$$p[\sigma^*] \in \left(0, \frac{z}{m}\right] \setminus \left\{\frac{z}{2m}\right\}. \tag{18}$$

Compared to the "anything goes" benchmark case resulting in the set $\mathcal{P}$, "almost anything goes" for our model of affine supply function competition. More specifically, only the prices

$$p_{asym} = \frac{zc}{N+cm} \text{ and } p\left(-\frac{m}{N}\right) = \frac{z(N+cm)}{m(2N+cm)}$$

in $\mathcal{P}$ cannot be generated through any symmetric Nash equilibrium. The price p_{asym} cannot be reached because it requires $\beta = \infty$; the price $p\left(-\frac{m}{N}\right) = \frac{z(N+cm)}{m(2N+cm)}$ is excluded because at the point $\beta = -\frac{m}{N}$ the discontinuous market-price function drops, by definition, to zero.

Proof of Proposition 1. Step 1. By Observation 2, $p(\beta)$ is strictly decreasing on $\left(-\infty, \beta_{pole}\right)$. As it is one-to-one and onto, the image of $\mathcal{B}_1$ under $p(\beta)$ thus becomes

$$p(\mathcal{B}_1) = \left[p\left(\beta^1\right), p_{asym}\right).$$

Direct substitution yields

$$\begin{aligned} p(\beta_1) &= \frac{z\left[(N-1)c\left(-\frac{1}{N-1}\left(m+\frac{2}{c}\right)\right)+1+cm\right]}{(N-1)(N+cm)\left(-\frac{1}{N-1}\left(m+\frac{2}{c}\right)\right)+m(N+1+cm)} \\ &= \frac{zc}{2N+cm} \end{aligned}$$

so that

$$p(\mathcal{B}_1) = \left[\frac{zc}{2N+cm}, \frac{zc}{N+cm}\right).$$

Step 2. Analogously, the image of $\mathcal{B}_2$ under $p(\beta)$ becomes

$$p(\mathcal{B}_2) = \left(p_{asym}, p\left(\beta^2\right)\right] \setminus \left\{p\left(-\frac{m}{N}\right)\right\}.$$

Direct substitution yields

$$p\left(\beta^2\right) = \frac{z\left[(N-1)c\left(-\frac{m}{N-1}\right)+1+cm\right]}{(N-1)(N+cm)\left(-\frac{m}{N-1}\right)+m(N+1+cm)} = \frac{z}{m}$$

as well as

$$p\left(-\frac{m}{N}\right) = \frac{z\left[(N-1)c\left(-\frac{m}{N}\right)+1+cm\right]}{(N-1)(N+cm)\left(-\frac{m}{N}\right)+m(N+1+cm)} = \frac{z(N+cm)}{m(2N+cm)},$$

implying

$$p(\mathcal{B}_2) = \left(\frac{zc}{N+cm}, \frac{z}{m}\right] \setminus \left\{\frac{z(N+cm)}{m(2N+cm)}\right\}.$$

Step 3. Because of

$$\begin{aligned} p(\mathcal{B}) &= p(\mathcal{B}_1) \cup p(\mathcal{B}_2) \\ &= \left[\frac{zc}{2N+cm}, \frac{zc}{N+cm}\right) \cup \left(\frac{zc}{N+cm}, \frac{z}{m}\right] \setminus \left\{\frac{z(N+cm)}{m(2N+cm)}\right\}, \end{aligned}$$

we obtain the desired result

$$p(\mathcal{B}) = \left[\frac{zc}{2N+cm}, \frac{z}{m}\right] \setminus \left\{\frac{zc}{N+cm}, \frac{z(N+cm)}{m(2N+cm)}\right\}.$$

□□

6 Competitive Nash equilibria

At the largest possible equilibrium price

$$\max p(\mathcal{B}) = p\left(\beta^2\right) = \frac{z}{m},$$

the buyer will buy zero units and the sellers make zero-profit. The according Nash equilibrium (which also consists of mutually worst responses) $\sigma^* = (a^*, \beta^*)_{i=1,..,N}$ such that[9]

$$\begin{aligned} a^* &= -\frac{z}{(N-1)}, \\ \beta^* &= \beta^2 = -\frac{m}{N-1} \end{aligned}$$

corresponds thus to a no-trade outcome which is the worst possible Nash equilibrium outcome for our model.

In contrast, the smallest possible equilibrium price

$$\min p(\mathcal{B}) = p\left(\beta^1\right) \tag{19}$$

would correspond to the best possible equilibrium outcome from the buyer's perspective. I am going to call any symmetric Nash equilibrium that generates (19) a *competitive* Nash equilibrium.

[9]We have

$$a^* = \frac{z\left[(cm-(N-2))\beta_2 - m + (N-1)c(\beta_2)^2\right]}{m + ((N-1)\beta_2 + m)(N+cm)},$$

which transforms for $\beta_2 = -\frac{m}{N-1}$ into

$$a^* = -\frac{z}{(N-1)}.$$

For the zero-cost model we have, by (18), that $\min p(\mathcal{B})$ does not exist but that

$$\inf p(\mathcal{B}) = 0$$

whereby

$$\inf p(\mathcal{B}) = p(\infty) = \frac{z}{(N-1)(N\infty + m(N+1)} = 0.$$

That is, if the slope of the equilibrium supply function approaches infinity, the sellers' price and profits approaches zero and the total supply sold to the buyer approaches $D(0) = z$. Although there does not exist a competitive Nash equilibrium for the zero-cost model, it can be approximated if the slope of the equilibrium supply function goes to plus or minus infinity.

The next proposition states that the situation is very different for non-zero quadratic costs.

Proposition 2. *For $c > 0$ there exists an unique competitive Nash equilibrium $\sigma^C = \left(a^C, \beta^C\right)_{i=1,..,N}$ with*

$$\begin{aligned} a^C &= -\frac{z}{(N-1)}, \\ \beta^C &= \beta_1 = -\frac{1}{N-1}\left(m + \frac{2}{c}\right) \end{aligned}$$

that generates the minimal price

$$\min p(\mathcal{B}) = p\left(\beta^1\right) = \frac{zc}{2N + cm}.$$

This competitive Nash equilibrium results into zero profits for the sellers.

Proof of Proposition 2. For $c > 0$ we have

$$\begin{aligned} p(\infty) &= p_{asym} = \frac{zc}{N + cm} \\ &> \\ \min p(\mathcal{B}) &= p\left(\beta_1 = -\frac{1}{N-1}\left(m + \frac{2}{c}\right)\right) = \frac{zc}{2N + cm}. \end{aligned}$$

The minimal price is for $c > 0$ thus (i) bounded away from $\frac{zc}{N+cm} > 0$ and (ii) it is not approached through an infinite slope but it is exactly obtained at the negative slope

$$\beta^C = \beta^1 = -\frac{1}{N-1}\left(m + \frac{2}{c}\right). \tag{20}$$

By (11) from the **Sketch of the Proof of Theorem 1**, we know that the firms are making a zero-profit at this Nash equilibrium. Substituting (20) for β in

$$a^* = \frac{z\left[\left(cm - (N-2)\right)\beta - m + (N-1)\,c\,(\beta)^2\right]}{m + \left((N-1)\,\beta + m\right)(N + cm)}$$

yields after simplification

$$a^C = -\frac{z}{(N-1)}.$$

□□

The following example illustrates the geometry underlying Proposition 2.

Example 1 revisited. Let $N = 2$, $z = 10$, $m = 1$, $c = 1$, to obtain the Möbius transformation

$$p(\beta) = \frac{10\beta + 20}{3\beta + 4}$$

from Example 1. The shaded area between the points

$$\beta^1 = -3 \text{ and } \beta^2 = -1$$

in Figure 2 corresponds to symmetric strategy-profiles that are mutually worst responses and therefore only Nash equilibria at the two points β^1 and β^2 (where they are also mutual best responses). The minimal price that can be generated by any symmetric Nash equilibrium is given as

$$p^{\min} = p\left(\beta^1 = -3\right) = 2,$$

which is strictly less than the price that could be approximated by an infinitely steep slope

$$\lim_{\beta \to \infty} p(\beta) = p_{asym} = \frac{10}{3}.$$

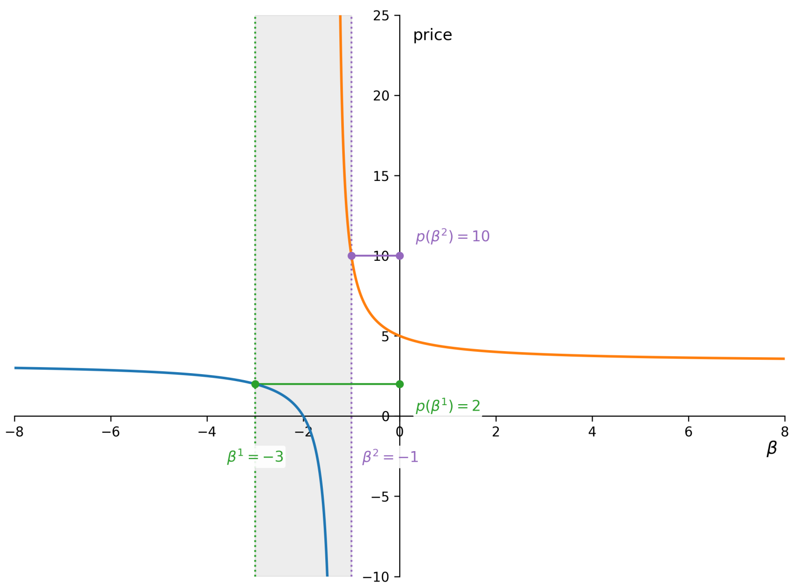

Figure 2: The minimal price and the maximal price of any symmetric Nash equilibrium for the Möbius transformation $p(\beta) = \frac{10\beta+20}{3\beta+4}$

Further note that we have for the maximal equilibrium price

$$p^{\max} = p\left(\beta^2 = -1\right) = 10$$

as well as

$$p\left(-\frac{1}{2}\right) = 6.$$

Consequently, the set of possible equilibrium prices becomes for this example the interval $\mathcal{P} = \left[p\left(\beta^1\right), p\left(\beta^2\right)\right] = [2, 10]$ minus the two points $p_{asym} = \frac{10}{3}$ and $p = 6.\square$

The important qualitative insight from Proposition 2 is that the minimal price $p^{\min}$ is generated by symmetric Nash equilibria whose supply function is strictly decreasing and has a rather steep slope. This is illustrated in the following example.

Example 2. Let $N = 2$, $z = 10$, $m = 1$ to obtain, in dependence on $c > 0$, the following competitive Nash equilibria $\left(a^C, \beta^C\right)_{i=1,..,N}$

$$\begin{aligned} a^C &= -10, \\ \beta^C &= \beta^1 = -\left(1 + \frac{2}{c}\right). \end{aligned}$$

which generate the minimal price

$$p^{\min} = p\left(\beta^1\right) = \frac{10 + 2a^C}{1 + 2\beta^C} = \frac{10c}{c+4}.$$

For $c = 1, 2, 3, 4, 5$ we obtain

c	a^C	β^C	$p^{\min}$	$D\left(p^{\min}\right)$
1	-10	-3	2	8
2	-10	-2	$\frac{10}{3}$	$\frac{20}{3}$
3	-10	$-\frac{5}{3}$	$\frac{30}{7}$	$\frac{40}{7}$
4	-10	$-\frac{3}{2}$	5	5
5	-10	$-\frac{7}{5}$	$\frac{50}{9}$	$\frac{40}{9}$

I leave it to the reader to verify that the profit of each seller i

$$\pi_i = p^{\min} \frac{D\left(p^{\min}\right)}{2} - \frac{c}{2}\left(\frac{D\left(p^{\min}\right)}{2}\right)^2$$

from the competitive Nash equilibria is indeed zero. Figure 3 plots the resulting aggregate supply functions, $TS(p) = S_1(p) + S_1(p)$, from these competitive Nash equilibria against the demand function $D(p) = 10 - p$.$\square$

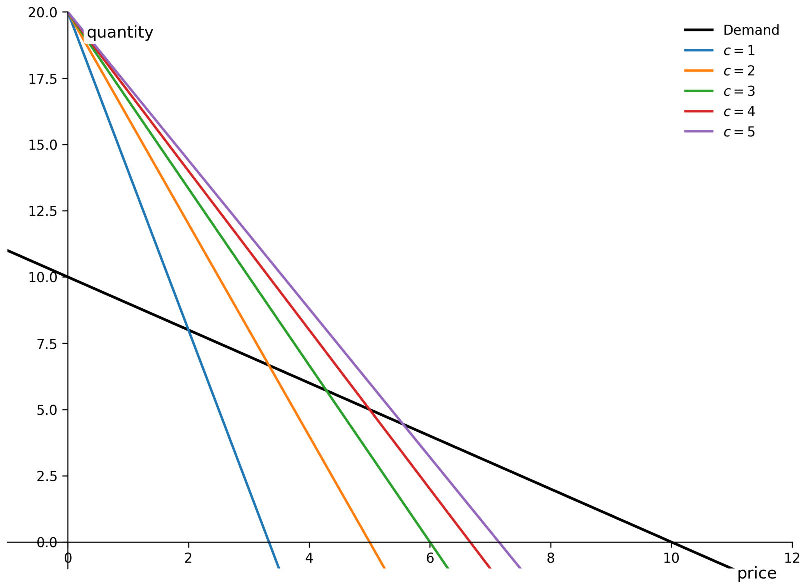

Figure 3: Generating the minimal price through competitive Nash equilibria for different cost parameters

7 Collusive Nash equilibria

I speak of a *collusive* Nash equilibrium iff this equilibrium generates a market-clearing price which coincides with the monopoly price. Denote by q the quantity a monopoly would choose to achieve the market price

$$p = \frac{z - q}{m}.$$

The profit-maximizing quantity q^M is then pinned down through the FOC

$$\frac{d}{dq}\left(\frac{z-q}{m}\right)q - \frac{c}{2}q^2\bigg|_{q^M} = 0$$

as

$$q^M = \frac{z}{2 + mc}$$

with corresponding monopoly price

$$p^M = \left(\frac{1 + mc}{m}\right)\left(\frac{z}{2 + mc}\right). \tag{21}$$

A symmetric Nash equilibrium $\sigma^M = \left(a^M, \beta^M\right)_{i=1,..,N}$ from Theorem 1 generates the monopoly price p^M iff we have for the extreme-point price function (15)

$$\begin{aligned} p\left(\beta^M\right) &= p^M \Leftrightarrow \\ \frac{z\left[(N-1)c\beta^M + 1 + cm\right]}{(N-1)(N+cm)\beta^M + m(N+1+cm)} &= \left(\frac{1+mc}{m}\right)\left(\frac{z}{2+mc}\right) \end{aligned} \tag{22}$$

provided that $\beta^M \in \mathcal{B}$. Because $p(\beta)$ is one-to-one, there can exist at most one β^M that generates the monopoly price. Solving (22) for β^M yields

$$\beta^M = -\frac{m(1+cm)}{N+(N-1)cm}. \tag{23}$$

Consider at first the zero-cost case $c = 0$. Then (23) becomes

$$\beta^M = -\frac{m}{N},$$

which is excluded from $\mathcal{B}$. I deal with this case in Remark 2 below.

Consider now the case $c > 0$. Note that

$$\begin{aligned} \beta^M &< -\frac{m}{N} \Leftrightarrow \\ -\frac{m(1+cm)}{N+(N-1)cm} &< -\frac{m}{N} \Leftrightarrow \\ 0 &< 1 \end{aligned}$$

is always satisfied. Next note that we also always have

$$\begin{aligned} -\frac{m}{N-1} &< \beta^M \Leftrightarrow \\ -\frac{m}{N-1} &< -\frac{m(1+cm)}{N+(N-1)cm} \Leftrightarrow \\ 0 &< m. \end{aligned}$$

Consequently, for $c > 0$ we have

$$\begin{aligned} \beta^M &= -\frac{m(1+cm)}{N+(N-1)cm} \in \left(-\frac{m}{N-1}, -\frac{m}{N}\right) \Rightarrow \\ \beta^M &\in \mathcal{B}. \end{aligned}$$

Substituting β^M in (2) and simplifying a^M yields the following result.

Proposition 3. *For $c > 0$ there exists an unique collusive Nash equilibrium $\sigma^M = \left(a^M, \beta^M\right)_{i=1,..,N}$ with*

$$\begin{aligned} a^M &= -\frac{z\left[Nc^2m^2 + 3Ncm + 2N - cm\right]}{N(2+cm)\left(N+(N-1)cm\right)}, \\ \beta^M &= -\frac{m(1+cm)}{N+(N-1)cm} \in \left(-\frac{m}{N-1}, -\frac{m}{N}\right) \end{aligned}$$

that generates the monopoly price

$$p^M = \left(\frac{1+mc}{m}\right)\left(\frac{z}{2+mc}\right).$$

Example 3. Let $N = 2$, $z = 10$, $m = 1$ to obtain, in dependence on $c > 0$, the following collusive Nash equilibria $\sigma^M = \left(a^M, \beta^M\right)_{i=1,..,N}$

$$\begin{aligned} a^M &= -10\frac{(2c^2+5c+4)}{2(2+c)^2}, \\ \beta^M &= -\left(\frac{1+c}{2+c}\right) \in \left(-1, -\frac{1}{2}\right). \end{aligned}$$

For values of $c = 1, 2, 3, 4, 5$ we obtain

c	a^M	β^M	$p\left[\sigma^M\right] = \frac{10+2a^M}{1+2\beta^M}$	$p^M = \left(\frac{1+c}{2+c}\right)\cdot 10$
1	$-\frac{55}{9}$	$-\frac{2}{3}$	$\frac{20}{3}$	$\frac{2}{3}\cdot 10$
2	$-\frac{55}{8}$	$-\frac{3}{4}$	$\frac{15}{2}$	$\frac{3}{4}\cdot 10$
3	$-\frac{37}{5}$	$-\frac{4}{5}$	8	$\frac{4}{5}\cdot 10$
4	$-\frac{70}{9}$	$-\frac{5}{6}$	$\frac{25}{3}$	$\frac{5}{6}\cdot 10$
5	$-\frac{395}{9}$	$-\frac{6}{7}$	$\frac{60}{7}$	$\frac{6}{7}\cdot 10$

which confirms that the price $p\left[\sigma^M\right]$ generated by these collusive Nash equilibria σ^M coincides with the respective monopoly price p^M. Figure 4 plots the resulting aggregate supply functions, $TS(p) = S_1(p) + S_1(p)$, from these Nash equilibria against the demand function $D(p) = 10 - p$.$\square$

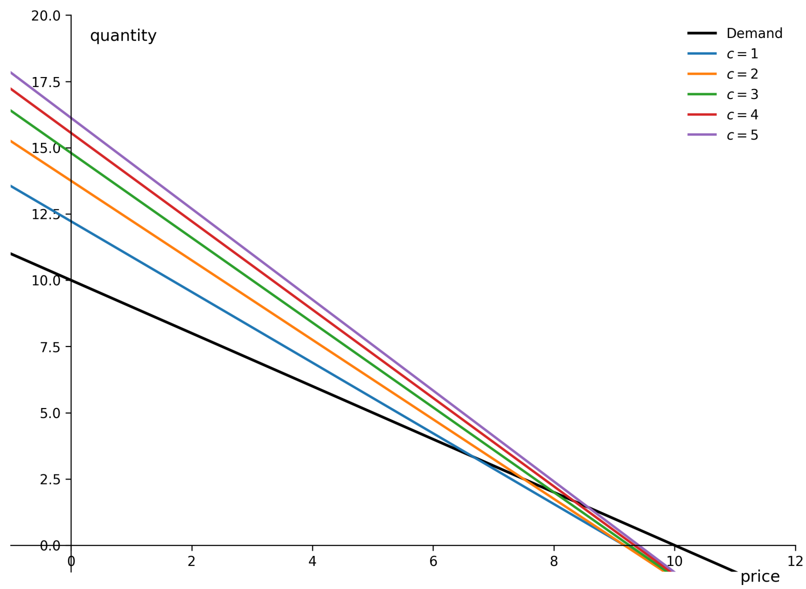

Figure 4: Generating the monopoly price through collusive Nash equilibria for different cost parameters

By Proposition 3, collusion always requires strictly decreasing supply functions with a moderately steep slope. Suppose that some regulator might be able to restrict the set of admissible supply functions to nondecreasing supply functions only. As the 'extreme-point' price function $p(\beta)$ is strictly decreasing for $\beta \geq 0$, the highest price that could be achieved in any symmetric Nash equilibrium would then coincide with the equilibrium price from Cournot output competition (cf. Corollary 2) given as

$$p(\beta = 0) = \frac{z[1 + cm]}{m(N + 1 + cm)}.$$

This brute-force regulation would thus effectively prevent collusion as a Nash equilibrium outcome in our economy.

Remark 2. Approximate collusion for $c = 0$. For $c = 0$, the equation (22) has the unique solution

$$\beta^M = -\frac{m}{N} \notin \mathcal{B},$$

which does not constitute a Nash equilibrium because it would result, for our discontinuous market-price function, in a zero market-price. However, as long as all sellers choose

$\beta^M(\varepsilon) = -\frac{m}{N} + \varepsilon$ for any $\varepsilon > 0$, this choice of $\beta^M(\varepsilon)$—with corresponding $a^M(\varepsilon)$—yields a Nash equilibrium from Theorem 1 because $\beta^M(\varepsilon) \in \mathcal{B}$. All sellers could thus coordinate on some Nash equilibrium $\sigma^M(\varepsilon)$ generating a market price

$$p\left[\sigma^M(\varepsilon)\right] = \frac{z + Na^M(\varepsilon)}{m + N\beta^M(\varepsilon)}$$

that gets, by continuity, for small $\varepsilon > 0$ arbitrarily close to the monopoly price p^M. This yields the following approximate collusion result, which appears already (without any formal details) as Robson's (1981, p.78) "limiting argument" for his zero-cost model.

Corollary 5. *Let $c = 0$. The symmetric Nash equilibrium $\sigma^M(\varepsilon) = \left(a^M(\varepsilon), \beta^M(\varepsilon)\right)_{i=1,..,N}$ with*

$$\begin{aligned}\beta^M(\varepsilon) &= -\frac{m}{N} + \varepsilon \\ a^M(\varepsilon) &= -\frac{z\left[2m + N(N-2)\varepsilon\right]}{N\left[2m + N(N-1)\varepsilon\right]}\end{aligned}$$

for $\varepsilon > 0$ generates a market price

$$p\left[\sigma^M(\varepsilon)\right] = \frac{z + Na^M(\varepsilon)}{m + N\beta^M(\varepsilon)} = \frac{z}{2m + N(N-1)\varepsilon}$$

which gets arbitrarily close to the monopoly price

$$p^M = \frac{z}{2m}$$

if $\varepsilon \to 0$.

8 Concluding remarks

This paper provides an in-depth analysis of the symmetric Nash equilibria for strategic games of affine supply function competition in which the strategic sellers have an identical quadratic cost function and face a strictly decreasing deterministic affine demand function. As a first main result, I have characterized the set of all symmetric Nash equilibria under the assumption that the sellers can choose arbitrarily increasing or decreasing affine supply functions. As a second main result, I have characterized the set of all market-clearing prices that can be generated by these symmetric Nash equilibria.

For non-zero quadratic costs there always exists a unique competitive Nash equilibrium—which is optimal from the buyer's perspective—as well as a unique collusive Nash equilibrium—which is optimal from the sellers' perspective. The interesting finding is that both opposite types of Nash equilibria require strictly decreasing supply functions whereby the slope of the respective equilibrium supply function must be steeper for the competitive than for the collusive Nash equilibrium.

Appendix: Proof of Theorem 1 (additional details)

Step 1. Fix any $(\sigma_i, \sigma_{-i}) = (a_i, A_{-i}, \beta_i, B_{-i})$ such that $m + \beta_i + B_{-i} \neq 0$. Rewrite $S_i[\sigma_i](p[\sigma])$ simply as

$$q_i = -a_i + \beta_i p$$

with

$$p = \frac{z + a_i + A_{-i}}{\beta_i + B_{-i} + m}$$

so that the rewritten profit (1) for a fixed (σ_i, σ_{-i}) becomes

$$\pi_i = pq_i - \frac{c}{2}q_i^2.$$

Any best response of i against (A_{-i}, B_{-i}) must then satisfy the two FOCs

$$\frac{\partial \pi_i}{\partial a_i} = \frac{d\pi_i}{dq_i}\frac{\partial q_i}{\partial a_i} = 0, \tag{24}$$

$$\frac{\partial \pi_i}{\partial \beta_i} = \frac{d\pi_i}{dq_i}\frac{\partial q_i}{\partial \beta_i} = 0. \tag{25}$$

Observe that

$$\frac{\partial q_i}{\partial \beta_i} = p + \beta_i \frac{\partial p}{\partial \beta_i} = \frac{z + a_i + A_{-i}}{\beta_i + B_{-i} + m} - \beta_i \frac{z + a_i + A_{-i}}{(\beta_i + B_{-i} + m)^2}$$

and

$$\begin{aligned} p\frac{\partial q_i}{\partial a_i} &= \left(\frac{z + a_i + A_{-i}}{\beta_i + B_{-i} + m}\right)\frac{\partial}{\partial a_i}\left(-a_i + \beta_i \frac{z + a_i + A_{-i}}{\beta_i + B_{-i} + m}\right) \\ &= \frac{z + a_i + A_{-i}}{(\beta_i + B_{-i} + m)}\left(-1 + \frac{\beta_i}{\beta_i + B_{-i} + m}\right) \\ &= (-)\frac{\partial q_i}{\partial \beta_i} \end{aligned}$$

Consequently, the first FOC (24) implies the second FOC (25) because of

$$\begin{aligned} \frac{d\pi_i}{dq_i}\frac{\partial q_i}{\partial a_i} &= 0 \Rightarrow \\ \frac{d\pi_i}{dq_i}(-)\,p\frac{\partial q_i}{\partial a_i} &= \frac{d\pi_i}{dq_i}\frac{\partial q_i}{\partial \beta_i} = 0. \end{aligned} \tag{26}$$

Whenever $p \neq 0$, the second FOC (25) also implies the first FOC (24).

Step 2. Focus on the first FOC (24). Rewrite the according first-order derivative as

$$\begin{aligned}\frac{\partial \pi_i}{\partial a_i} &= \left(\frac{dp}{dq_i}q_i + p - cq_i\right)\frac{\partial q_i}{\partial a_i}\\ &= \left((-)\frac{1}{B_{-i}+m}q_i + p - cq_i\right)\frac{\partial q_i}{\partial a_i},\end{aligned}$$

whereby $\frac{dp}{dq_i} = (-)\frac{1}{m+B_{-i}}$ follows from

$$\begin{aligned}z - mp &= q_i - A_{-i} + B_{-i}p \Rightarrow\\ p(q_i) &= \frac{z - q_i + A_{-i}}{B_{-i}+m}.\end{aligned}$$

Substituting

$$\begin{aligned}p &= \frac{z + a_i + A_{-i}}{\beta_i + B_{-i} + m},\\ q_i &= \frac{-a_i(B_{-i}+m) + \beta_i(z + A_{-i})}{\beta_i + B_{-i} + m},\\ \frac{\partial q_i}{\partial a_i} &= (-)\left(\frac{B_{-i}+m}{\beta_i + B_{-i} + m}\right)\end{aligned}$$

yields the first-order derivative

$$\begin{aligned}\frac{\partial \pi_i}{\partial a_i} = &\left[(-)\left(2(B_{-i}+m) + c(B_{-i}+m)^2\right)a_i\right.\\ &\left.+\left(\beta_i - (B_{-i}+m) + c\beta_i(B_{-i}+m)\right)(A_{-i}+z)\right]\\ &\left(\frac{1}{(\beta_i + B_{-i} + m)^2}\right).\end{aligned} \tag{27}$$

Step 3. The first-order derivative (27) yields the following expression for the first FOC (24):

$$\begin{aligned}0 = &(-)\left(2(B_{-i}+m) + c(B_{-i}+m)^2\right)a_i^*\\ &\left(\beta_i - (B_{-i}+m) + c\beta_i(B_{-i}+m)\right)(A_{-i}+z).\end{aligned} \tag{28}$$

For any symmetric strategy-profile $(a^*, \beta^*)_{i=1,..,N}$, the FOC (28) becomes

$$\begin{aligned}0 = &\left[(-)\left(2((N-1)\beta^* + m) + c((N-1)\beta^* + m)^2\right)a^*\right.\\ &\left(\beta^* - ((N-1)\beta^* + m) + c\beta^*((N-1)\beta^* + m)\right)(N-1)a^*\\ &\left.+\left(\beta^* - ((N-1)\beta^* + m) + c\beta^*((N-1)\beta^* + m)\right)z\right].\end{aligned}$$

$$\Leftrightarrow$$

$$a^* = \frac{z\left[(cm-(N-2))\,\beta^* - m + (N-1)\,c\,(\beta^*)^2\right]}{m+((N-1)\,\beta^*+m)\,(N+cm)}.$$

We have thus a symmetric Nash equilibrium $(a^*, \beta^*)_{i=1,..,N}$ iff the (a^*, β^*) are utility-maximizers.

Step 4. The FOC (28) can be rewritten as the following function of a_i^* in dependence of β_i

$$a_i^*(\beta_i) = \frac{\beta_i - (B_{-i}+m) + c\beta_i\,(B_{-i}+m)}{2\,(B_{-i}+m) + c\,(B_{-i}+m)^2}\,(A_{-i}+z)\,,$$

provided that $2\,(B_{-i}+m) + c\,(B_{-i}+m)^2 \neq 0$. For this function the constant utility property (6), i.e.,

$$\frac{d\pi_i}{d\beta_i}\,(a_i^*(\beta_i)\,,\beta_i) = 0,$$

is satisfied for all $\beta_i \in \mathcal{D}$. This implies that the SOC for a (global) utility maximum at $(a_i^*(\beta_i)\,,\beta_i)$ holds iff

$$\frac{\partial^2\pi_i}{\partial a_i\partial a_i}\,(a_i^*(\beta_i)\,,\beta_i) < 0$$

for $\beta_i \in \mathcal{D}$.

Step 5. Differentiating (27) yields

$$\frac{\partial^2\pi_i}{\partial a_i\partial a_i} = (-)\left(2\,(B_{-i}+m) + c\,(B_{-i}+m)^2\right)\left(\frac{1}{(\beta_i+B_{-i}+m)^2}\right).$$

If

$$B_{-i}+m > 0, \tag{29}$$

then

$$\frac{\partial^2\pi_i}{\partial a_i\partial a_i} < 0 \tag{30}$$

holds iff

$$B_{-i}+m > -\frac{2}{c},$$

which is always satisfied under (29). For a symmetric strategy-profile $(a^*, \beta^*)_{i=1,..,N}$ this means that (30) always holds if

$$\beta^* > -\left(\frac{m}{N-1}\right) = \beta^2.$$

If we have instead

$$B_{-i} + m < 0, \tag{31}$$

then (30) holds iff

$$B_{-i} + m < -\frac{2}{c},$$

which implies (31). For a symmetric Nash equilibrium $(a^*, \beta^*)_{i=1,..,N}$ the condition (30) thus also holds whenever

$$\beta^* < -\frac{1}{N-1}\left(m + \frac{2}{c}\right) = \beta^1.$$

To summarize: for any symmetric strategy-profile $(a^*, \beta^*)_{i=1,..,N}$ the SOC (30) for a utility maximum holds iff

$$\beta^* \in \left(\left(-\infty, \beta^1\right) \cup \left(\beta^2, \infty\right)\right) \cap \mathcal{D}.$$

.

Step 6. If either $\beta_j = \beta^1$ for all $j \neq i$ or $\beta_j = \beta^2$ for all $j \neq i$, we have

$$\begin{aligned} 2\left(B_{-i} + m\right) + c\left(B_{-i} + m\right)^2 &= 0 \Rightarrow \\ 2\left(\left(N-1\right)\beta_j + m\right) + c\left(\left(N-1\right)\beta_j + m\right)^2 &= 0. \end{aligned}$$

This case—where utility maximization and minimization coincide—has been discussed in detail in The Sketch of the Proof of Proposition 1 with the result that

$$\beta^1, \beta^2 \in \mathcal{B}.$$

Step 7. Finally, note that we have for symmetric strategy-profiles

$$\mathcal{D} = \mathbb{R} \backslash \left\{-\frac{m}{N}, -\frac{m\left(N+1+cm\right)}{\left(N-1\right)\left(N+cm\right)}\right\}.$$

The function $a\left(\beta^*\right)$ is not well-defined for $\beta^* = -\frac{m}{N}$ because of the discontinuity of π_i at

$$\beta_i + B_{-i} + m = 0 \Rightarrow N\beta^* + m = 0.$$

Moreover, the function $a\left(\beta^*\right)$ is also not well-defined at

$$\beta^* = \beta_{pole} = -\frac{m\left(N+1+cm\right)}{\left(N-1\right)\left(N+cm\right)}$$

because it's denominator would become zero at this point. As it is, this point will turn out to be the unique pole β_{pole} of the Möbius transformation $p(\beta)$ defined by (15). I will come back to β_{pole} in Observation 2 which states that we always have

$$\beta^1 < \beta_{pole} < \beta^2,$$

implying the desired result

$$\begin{aligned} \mathcal{B} &= \left(\left(-\infty, \beta^1\right] \cup \left[\beta^2, \infty\right)\right) \cap \mathcal{D} \\ &= \left(-\infty, \beta^1\right] \cup \left[\beta^2, \infty\right) \setminus \left\{-\frac{m}{N}\right\}. \end{aligned}$$

□□

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