Resolving the friction between Turnbull's (1983) and Klemperer-Meyer's (1989) Nash equilibria in affine supply functions

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Abstract

Turnbull (1983) describes unique symmetric Nash equilibria for supply function competition under demand uncertainty which consist of affine strategies that are nonlinear. For the same economy Klemperer and Meyer (1989) derive unique Nash equilibria that are linear. This note shows that the friction between both results comes from a mathematical error in Turnbull (1983). Correcting this error establishes the same linear Nash equilibria for both models.

Keywords: Supply function competition; Demand uncertainty; Uniqueness of Nash equilibria

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1 Introduction

The introduction of demand uncertainty can drastically reduce the multiplicity of Nash equilibria for models of supply function competition. The following—informal—argument about this possible impact of demand uncertainty goes back to Klemperer and Meyer (1989). Under demand certainty firms know their residual demand curve in an equilibrium so that the choice of a supply function corresponds to the choice of a single utility maximizing price-output pair on the residual demand curve. The flexibility of supply functions might now allow for the support of many different price-output pairs as mutually best responses. In contrast, under demand uncertainty the choice of a supply function will result for different states of the world in different price-output pairs. As utility maximization does thus no longer concern just single points on a known residual demand function, the choice of an (expected) utility maximizing supply function comes with an additional role that may result in the reduction of Nash equilibria.

Of course, the details of how (or whether) this informal argument works out for a specific game of supply function competition will strongly depend on modelling choices concerning, e.g., (i) the sets of admissible supply functions, (ii) the specifications of cost functions, and (iii) the kind of considered demand uncertainty.[1] For instance, for admissible strategies restricted to affine supply functions[2], Robson (1981) establishes the existence of multiple symmetric Nash equilibria under demand certainty whereas existence fails under demand uncertainty. Turnbull (1983) (=T83) implies the remarkable result that adding quadratic costs to Robson's model keeps the multiplicity of equilibria under demand certainty but gives rise to a unique symmetric Nash equilibrium under any kind of (horizontal) demand uncertainty. Klemperer and Meyer (1989) (=KM89) show that a large continuum of nonlinear Nash equilibria can be reduced to a single linear Nash equilibrium if the demand uncertainty term has full support on the positive

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[1] The picture becomes more complicated if uncertainty about the residual demand function additionally results from players' different private signals. For the Bayesian game of demand function competition in Kyle (1989) unique symmetric linear Bayesian Nash equilibria exist only under noisy demand by the liquidity trader whereas they fail to exist for noise-free demand. In Vives's (2011) Bayesian game of supply function competition symmetric affine Nash equilibria are unique if the liquidity trader's demand is noise-free.

[2] For this paper's analysis it is crucial to distinguish between supply functions $S(p) = -\alpha + \beta p$ that are *affine* in price p and the special case of *linear* supply functions $S(p) = \beta p$, i.e., affine supply functions that go through the origin.

real line. KM89 restrict their equilibrium analysis to twice continuously differentiable supply functions starting at the origin. This ad hoc restriction rules out any Nash equilibria in affine supply functions which do not reduce to Nash equilibria in linear supply functions. In contrast, the unique symmetric Nash equilibria in T83 consist of affine supply functions that do not reduce to linear functions.

This paper revisits T83's original equilibrium analysis. If T83's characterization of unique symmetric Nash equilibria is correct, KM89's unique symmetric Nash equilibria in linear strategies could only exist if affine nonlinear strategies are excluded as admissible strategies. But such ad hoc exclusion of affine supply functions would question the generality of the equilibrium analysis in KM89. So, something has to give here: either some mathematical error has occurred, or the unique Nash equilibria in KM89 fail to exist if nonlinear affine supply functions become admissible. If the latter was the case, there would be implications for the interpretation of the applied and empirical work that is based on KM89's characterization of Nash equilibria.[3]

As main result I show that T83's original characterization of unique symmetric Nash equilibria is incorrect and that the correct Nash equilibria for his model with affine supply functions under arbitrary demand uncertainty coincide indeed with the unique linear equilibria derived in KM89. Although the friction between T83 and KM89 is thus resolved by correcting a mathematical error in T83, Turnbull's original approach adds nevertheless an important additional insight to KM89. Namely, whereas KM89 exclude nonlinear affine strategies as possible equilibrium choices in an ad hoc way, the present paper's corrected analysis of T83 shows that the linear Nash equilibria of KM89 also obtain as unique equilibria without the ad hoc exclusion of any nonlinear affine Nash equilibria.

The remainder of this analysis proceeds as follows. Section 2 introduces the economy. Turnbull's original analysis is revisited—and corrected—in Section 3. Section 4 concludes with a discussion about the relationship between T83 and KM89.

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[3] KM89's characterization of Nash equilibria has been heavily used in applied investigations of electricity markets (e.g., Green and Newbery 1992; Bolle 1992; Holmberg and Newbery 2010).

2 Affine supply function competition

I generalize the duopoly set-up of T83 and KM89 to $N \geq 2$ firms, i.e., players. Negative prices and negative amounts of supply are admissible in my model.[4] My notation is, basically, a mixture between T83 and KM89.

2.1 Market demand

Fix some probability space $(\Omega, \mathcal{F}, \mu)$. In state $\omega \in \Omega$ the market demand at price $p \in \mathbb{R}$ is given as

$$D\left(p, \boldsymbol{\varepsilon}\left(\omega\right)\right) = \boldsymbol{\varepsilon}\left(\omega\right) - mp$$

for some fixed parameter value $m > 0$. The $\mathcal{F}$-measurable function $\boldsymbol{\varepsilon} : \Omega \to \mathbb{R}$ satisfies (i) $\boldsymbol{\varepsilon}\left(\omega\right) \geq 0$ for all $\omega \in \Omega$ and (ii)

$$0 < \int_{\omega \in \Omega} \left(\boldsymbol{\varepsilon}\left(\omega\right)\right)^2 d\mu < \infty.$$

There is *demand certainty* if $\mu\left(\boldsymbol{\varepsilon} = \varepsilon\right) = 1$ for some $\varepsilon > 0$ and *demand uncertainty* else.

2.2 Players, strategies

Player $i \in \{1, ..., N\}$, $N \geq 2$, chooses an individual strategy which corresponds to the submission of a supply-schedule to the Walrasian auctioneer. Player i's set of individual strategies is given as

$$\Sigma_i = \mathbb{R} \times \mathbb{R}_{>0}$$

with generic element $\sigma_i = (a_i, \beta_i)$. If i chooses the individual strategy $\sigma_i = (a_i, \beta_i)$, his supply function $S_i\left[\sigma_i\right] : \mathbb{R} \to \mathbb{R}$ takes on the affine structure

$$S_i\left[\sigma_i\right](p) = -a_i + \beta_i p.$$

By choosing strategy $\sigma_i \in \Sigma_i$, player i submits the supply-schedule

$$\left(S_i\left[\sigma_i\right](p), p\right)_{p \in \mathbb{R}}$$

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[4] That said, one main insight of my analysis will be that negative prices or negative amounts of supply are impossible in any symmetric Nash equilibrium under demand uncertainty.

to the Walrasian auctioneer. This submission commits i to supply the amount $S_i[\sigma_i](p)$ whenever the Walrasian auctioneer announces p as market-clearing price.

2.3 Market-clearing price

The price p clears markets in state $\omega \in \Omega$ under strategy profile

$$\sigma = (\sigma_1, ..., \sigma_N) \in \times_{i=1}^{N} \Sigma_i$$

iff the aggregate supply of all players meets the market demand; that is, iff

$$\sum_{i=1}^{N} S_i[\sigma_i](p) = D(p, \boldsymbol{\varepsilon}(\omega)) \Leftrightarrow \sum_{i=1}^{N} -a_i + \beta_i p = \boldsymbol{\varepsilon}(\omega) - mp.$$

The corresponding market price function $\mathbf{p}[\sigma] : \Omega \to \mathbb{R}$ under strategy profile $\sigma \in \Sigma$ is given as

$$\mathbf{p}[\sigma](\omega) = \frac{\sum_{i=1}^{N} a_i + \boldsymbol{\varepsilon}(\omega)}{\sum_{i=1}^{N} \beta_i + m}. \tag{1}$$

2.4 Expected profit and strategic game

I follow KM89 and assume that each player i faces a quadratic cost function in the supplied amount

$$\frac{c}{2}(\cdot)^2$$

with $c > 0$.[5] This specification corresponds to the special case of T83's cost function for which his additional cost parameters become $c_0 = c_1 = 0$.

Player i's expected profit in dependence on the chosen strategy profile is given as $\bar{\pi}_i : \Sigma \to \mathbb{R}$ such that

$$\begin{aligned} \bar{\pi}_i(\sigma_i, \sigma_{-i}) &= \int_{\omega\in\Omega} \mathbf{p}[\sigma](\omega) S_i[\sigma_i](\mathbf{p}[\sigma](\omega)) - \frac{c}{2}\left(S_i[\sigma_i](\mathbf{p}[\sigma](\omega))\right)^2 d\mu \qquad (2) \\ &= \int_{\omega\in\Omega} \left(\frac{a_i + \sum_{j\neq i} a_j + \boldsymbol{\varepsilon}(\omega)}{\beta_i + \sum_{j\neq i}\beta_j + m}\right)\left(-a_i + \beta_i \frac{a_i + \sum_{j\neq i} a_j + \boldsymbol{\varepsilon}(\omega)}{\beta_i + \sum_{j\neq i}\beta_j + m}\right) \\ &\quad - \frac{c}{2}\left(-a_i + \beta_i \frac{a_i + \sum_{j\neq i} a_j + \boldsymbol{\varepsilon}(\omega)}{\beta_i + \sum_{j\neq i}\beta_j + m}\right)^2 d\mu. \end{aligned}$$

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[5] For $c = 0$ we would run into existence problems under demand uncertainty (cf. Robson (1981) as well as the characterization of β in Proposition 2 which is not well-defined for $c = 0$).

Define the economy as the strategic game

$$\Gamma = \langle \Sigma_i, U_i \rangle_{i \in \{1,...,N\}}$$

such that i's expected utility simply coincides with his expected profits, that is, $U_i : \Sigma \to \mathbb{R}$ satisfies

$$U_i(\sigma) = \bar{\pi}_i(\sigma).$$

The strategy profile $\sigma^* \in \Sigma$ is a Nash equilibrium of Γ iff, for all $i \in \{1, ..., N\}$,

$$U_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq U_i\left(\sigma_i, \sigma_{-i}^*\right) \text{ for all } \sigma_i \in \Sigma_i.$$

3 Revisiting—and correcting—Turnbull (1983)

In T83 the random variable $\boldsymbol{\varepsilon}$ is given as

$$\boldsymbol{\varepsilon}(\omega) = d_0 + \boldsymbol{\eta}(\omega)$$

with $\int_{\omega \in \Omega} \boldsymbol{\eta}(\omega)\, d\mu = 0$. Demand uncertainty therefore holds iff the shock term has positive variance, i.e., iff

$$\sigma_\eta^2 = \int_{\omega \in \Omega} (\boldsymbol{\eta}(\omega))^2\, d\mu > 0.$$

Using T83's notation for expected values, the expected profit (2) of any $\sigma \in \Sigma$ can be equivalently transformed into

$$\bar{\pi}_i(\sigma_i, \sigma_{-i}) = \bar{P}\bar{q}_i - c(\bar{q}_i) + \frac{\left(\beta_i - \frac{c}{2}(\beta_i)^2\right)}{\left(\beta_i + \sum_{j \neq i} \beta_j + m\right)^2} \sigma_\eta^2$$

such that

$$\begin{aligned} \bar{q}_i &= \left(-a_i + \beta_i \frac{a_i + \sum_{j \neq i} a_j + d_0}{\beta_i + \sum_{j \neq i} \beta_j + m}\right), \\ \bar{P} &= \frac{d_0 - \left(\Sigma_{i=1}^N \bar{q}_i\right)}{m} = \frac{a_i + \sum_{j \neq i} a_j + d_0}{\beta_i + \sum_{j \neq i} \beta_j + m}, \\ c(\bar{q}_i) &= \frac{c}{2}\left(-a_i + \beta_i \frac{a_i + \sum_{j \neq i} a_j + d_0}{\beta_i + \sum_{j \neq i} \beta_j + m}\right)^2. \end{aligned}$$

Any best response of i is pinned down by the two FOCs

$$\begin{aligned}\frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{da_i} &= \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\bar{q}_i}\frac{d\bar{q}_i}{da_i} = \left(\frac{d\bar{P}}{d\bar{q}_i}\bar{q}_i + \bar{P} + c'\left(\bar{q}_i\right)\right)\frac{d\bar{q}_i}{da_i} = 0,\\ \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\beta_i} &= \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\bar{q}_i}\frac{d\bar{q}_i}{d\beta_i} + \frac{d}{d\beta_i}\frac{\left(\beta_i - \frac{c}{2}\left(\beta_i\right)^2\right)}{\left(\beta_i + \sum_{j\neq i}\beta_j + m\right)^2}\sigma_\eta^2 = 0.\end{aligned}$$

Next observe that

$$\begin{aligned}\bar{P}\frac{d\bar{q}_i}{da_i} &= -\left(\frac{a_i + \sum_{j\neq i}a_j + d_0}{\beta_i + \sum_{j\neq i}\beta_j + m}\right) + \left(\frac{\beta_i}{\beta_i + \sum_{j\neq i}\beta_j + m}\right)\left(\frac{a_i + \sum_{j\neq i}a_j + d_0}{\beta_i + \sum_{j\neq i}\beta_j + m}\right)\\ &= -\frac{d\bar{q}_i}{d\beta_i}.\end{aligned}$$

If there is no demand uncertainty, i.e., if $\sigma_\eta^2 = 0$, the above FOCs thus reduce to a single equation in a_i and β_i because of

$$\begin{aligned}\frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{da_i} &= \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\bar{q}_i}\frac{d\bar{q}_i}{da_i} = 0\\ &\Leftrightarrow\\ \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\beta_i} &= \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\bar{q}_i}\frac{d\bar{q}_i}{d\beta_i} = \frac{d\bar{\pi}_i\left(\sigma_i,\sigma_{-i}\right)}{d\bar{q}_i}\left(-\bar{P}\right)\frac{d\bar{q}_i}{da_i} = 0.\end{aligned}$$

This indeterminacy gives rise to Robson's (1981) and T83's "anything goes" result under demand certainty.[6]

Proposition 1. *If there is demand certainty, i.e.,* $\sigma_\eta^2 = 0$, *the symmetric Nash equilibria of* Γ *are the members of the set*

$$\left\{(a,\beta)_{i=1,..,N} \in \Sigma \mid a = \frac{d_0\left(cm - (N-2)\right)\beta - md_0 + d_0\left(N-1\right)c\beta^2}{m + \left((N-1)\beta + m\right)\left(N + cm\right)},\ \beta > 0\right\}.$$

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[6] The Nash equilibria of Proposition 1 also hold for the special case $c = 0$. These multiple Nash equilibria σ^* with $\sigma_i^* = (a,\beta)$ such that

$$a = \frac{d_0\left(-\left(N-2\right)\beta - m\right)}{\left(N\left(N-1\right)\beta + \left(N+1\right)m\right)},\ \beta > 0$$

had been originally derived in Robson (1981, Eq. (13)) who even allows for $\beta \leq 0$.

Proof. Consider σ_{-i} such that $a = a_j$ and $\beta = \beta_j$ for all $j \neq i$ so that the FOC becomes

$$\begin{aligned}
0 &= \frac{d\bar{\pi}_i\left(\sigma_i, \sigma_{-i}\right)}{d\bar{q}_i}\frac{d\bar{q}_i}{da_i} \\
&= \frac{1}{\beta_i + (N-1)\beta + m}\left(-a_i + \beta_i\frac{a_i + (N-1)a + d_0}{\beta_i + (N-1)\beta + m}\right) \\
&\quad + \left(-1 + \frac{\beta_i}{\beta_i + (N-1)\beta + m}\right)\left(\frac{a_i + (N-1)a + d_0}{\beta_i + (N-1)\beta + m}\right) \\
&\quad - c\left(-a_i + \beta_i\frac{a_i + (N-1)a + d_0}{\beta_i + (N-1)\beta + m}\right)\left(-1 + \frac{\beta_i}{\beta_i + (N-1)\beta + m}\right).
\end{aligned}$$

For any symmetric Nash equilibrium we have

$$\begin{aligned}
0 &= -a\left(N\beta + m\right) + \beta\left(Na + d_0\right) - \left((N-1)\beta + m\right)\left(Na + d_0\right) \\
&\quad - c\left(-a\left(N\beta + m\right) + \beta\left(Na + d_0\right)\right)\left((N-1)\beta + m\right) \\
&= -\left[N(N-1)\beta + m(N+1) + m\left((N-1)c\beta + cm\right)\right]a \\
&\quad + \beta d_0 - \left((N-1)\beta + m\right)d_0 + c\beta d_0\left((N-1)\beta + m\right)
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
a &= \frac{\beta d_0 - (N-1)\beta d_0 - md_0 + \left((N-1)cd_0\beta^2 + c\beta d_0 m\right)}{N(N-1)\beta + m(N+1) + m(N-1)c\beta + cm^2} \\
&= \frac{d_0\left(cm - (N-2)\right)\beta - md_0 + d_0(N-1)c\beta^2}{m + \left((N-1)\beta + m\right)\left(N + cm\right)}.
\end{aligned} \tag{3}$$

□□

Let me translate my results into T83's original notation given as follows.

T83	this paper
c_1	0
c_2	$\frac{c}{2}$
d_0	d_0
d_1	$-m$
α	$-a$
β	β

For the duopoly case $N = 2$ analyzed in T83 we obtain[7]

$$\begin{aligned} a &= \frac{(cmd_0)\,\beta - md_0 + d_0 c\beta^2}{m + (\beta + m)(2 + cm)} && (5) \\ &\Leftrightarrow \\ \alpha &= \frac{-(2c_2 d_1 d_0)\,\beta + d_1 d_0 + 2c_2 d_0 \beta^2}{d_1 + 2(1 - c_2 d_1)(d_1 - \beta)}. && (6) \end{aligned}$$

Turn now to the case of demand uncertainty.

Proposition 2. *If there is demand uncertainty, i.e.,* $\sigma_\eta^2 > 0$*, then there exists a unique symmetric Nash equilibrium* $(a, \beta)_{i=1,..,N} \in \Sigma$ *of* Γ *such that*

$$\begin{aligned} a &= 0, \\ \beta &= \frac{(N - 2 - cm)}{2c(N-1)} + \sqrt{\left(\frac{(N - 2 - cm)}{2c(N-1)}\right)^2 + \frac{m}{c(N-1)}}. \end{aligned}$$

Proof. Step 1. Recall that

$$\frac{d\bar{\pi}_i(\sigma_i, \sigma_{-i})}{d\bar{q}_i}\frac{d\bar{q}_i}{da_i} = 0 \text{ and } \bar{P}\frac{d\bar{q}_i}{da_i} = -\frac{d\bar{q}_i}{d\beta_i},$$

which implies, in an optimum,

$$\frac{d\bar{\pi}_i(\sigma_i, \sigma_{-i})}{d\bar{q}_i}\frac{d\bar{q}_i}{d\beta_i} = 0.$$

Consequently, the second FOC becomes for $\sigma_\eta^2 > 0$

$$\begin{aligned} \frac{d\bar{\pi}_i(\sigma_i, \sigma_{-i})}{d\beta_i} &= \frac{d\bar{\pi}_i(\sigma_i, \sigma_{-i})}{d\bar{q}_i}\frac{d\bar{q}_i}{d\beta_i} + \frac{d}{d\beta_i}\frac{\left(\beta_i - \frac{c}{2}(\beta_i)^2\right)}{\left(\beta_i + \sum_{j\neq i}\beta_j + m\right)^2}\sigma_\eta^2 \\ &= \frac{d}{d\beta_i}\frac{\left(\beta_i - \frac{c}{2}(\beta_i)^2\right)}{\left(\beta_i + \sum_{j\neq i}\beta_j + m\right)^2}\sigma_\eta^2 = 0. \end{aligned}$$

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[7] Note that my characterization (6) of α slightly differs (in the denominator) from T83's own derivation (cf. Eq. (12)) given as:

$$\alpha = \frac{-(2c_2 d_1 d_0)\,\beta + d_1 d_0 + 2c_2 d_0 \beta^2}{d_1 + 2(1 - 2c_2 d_1)(d_1 - \beta)}. \qquad (4)$$

While I am confident that my derivation of α is the correct one, it will actually be irrelevant for the result of Proposition 2 whether my (6) or T83's (4) is used as characterization for $\alpha = -a$ as only substitution of β into the (identical) nominator matters for both cases.

This pins down $\beta > 0$ in a symmetric Nash equilibrium through

$$\begin{aligned} 0 &= \frac{(1-c\beta)}{(N\beta+m)^2}\sigma_\eta^2 - 2\frac{\left(\beta_i - \frac{c}{2}(\beta_i)^2\right)}{(N\beta+m)^3}\sigma_\eta^2 \\ &= \beta^2 - \frac{(N-2-cm)}{c(N-1)}\beta - \frac{m}{c(N-1)} \end{aligned}$$

resulting in

$$\beta = \frac{(N-2-cm)}{2c(N-1)} + \sqrt{\left(\frac{(N-2-cm)}{2c(N-1)}\right)^2 + \frac{m}{c(N-1)}} > 0. \tag{7}$$

Step 2. To pin down the equilibrium value for a, we have to substitute (7) in (3). Focus on the nominator

$$(cmd_0 + (2-N)d_0)\beta - md_0 + d_0(N-1)c\beta^2$$

of (3) and observe that substituting

$$\beta^2 = \frac{(N-2-cm)}{c(N-1)}\beta + \frac{m}{c(N-1)}$$

in this nominator yields

$$\begin{aligned} & d_0(cm-(N-2))\beta - md_0 + d_0(N-1)c\beta^2 \\ = \;& d_0(cm-(N-2))\beta - md_0 + d_0(N-1)c\left(\frac{(N-2-cm)}{c(N-1)}\beta + \frac{m}{c(N-1)}\right) \\ = \;& -d_0(N-2-cm)\beta - md_0 + d_0(N-2-cm)\beta + d_0 m \\ = \;& 0. \end{aligned}$$

Consequently, we must have $a = 0$ as the denominator in (3) is strictly positive.□□

Let us compare Proposition 2 with T83's original result for $N = 2$. Observe at first that

$$\begin{aligned} \beta &= \frac{(-cm)}{2c} + \sqrt{\left(\frac{(-cm)}{2c}\right)^2 + \frac{m}{c}} = \frac{1}{2}\left(-m + \sqrt{m^2 + \frac{4m}{c}}\right) \\ &\Leftrightarrow \\ \beta &= \frac{d_1}{2} + \sqrt{\left(\frac{d_1}{2}\right)^2 - \frac{d_1}{2c_2}} = \frac{1}{2}\left(d_1 + \sqrt{d_1^2 - \frac{2d_1}{c_2}}\right), \end{aligned}$$

which is T83's original expression for β (cf. Eqs. (6) and (13)) as well as the expression derived in KM89 (Eq. (11)) for their linear example (for which $-a = \alpha = 0$ holds by ad hoc assumption). However, in contrast to our result

$$-a = \alpha = 0,$$

Turnbull claims that the substitution of

$$\beta = \frac{1}{2}\left(d_1 + \sqrt{d_1^2 - \frac{2d_1}{c_2}}\right),$$

given as the solution to

$$\beta^2 = d_1\beta - \frac{d_1}{2c_2}, \tag{8}$$

into (6) yields

$$\alpha = -\frac{1}{2}c_2\left(d_1 + \sqrt{d_1^2 - \frac{2d_1}{c_2}}\right) \neq 0. \tag{9}$$

One can directly verify that this claim is incorrect. Substituting (8) into the nominator

$$-\left(2c_2d_1d_0\right)\beta + d_1d_0 + 2c_2d_0\beta^2$$

of (6) shows that

$$\begin{aligned} & -\left(2c_2d_1d_0\right)\beta + d_1d_0 + 2c_2d_0\left(d_1\beta - \frac{d_1}{2c_2}\right) \\ = & -\left(2c_2d_1d_0\right)\beta + d_1d_0 + 2c_2d_0d_1\beta - d_0d_1 \\ = & \ 0, \end{aligned}$$

which implies $\alpha = 0$. Therefore, the characterization (9) with $-a = \alpha \neq 0$ in T83 must be incorrect.

4 Concluding discussion: relationship to Klemperer and Meyer (1989)

The unique Nash equilibria from T83's corrected analysis (Proposition 2) coincide with the unique linear Nash equilibria derived in KM89 for T83's duopoly economy (Eq.11 in KM89). However, the derivation in KM89 proceeds rather differently as these authors exclude ad hoc any equilibria in affine supply schedules that are not linear. More precisely, for interpretational reasons KM89 want to exclude the possibility that equilibrium prices or supplied amounts could take on negative values in some state of the world. To this purpose, they restrict attention to symmetric Nash equilibria σ^* such that $S_i[\sigma_i^*](0) = 0$ (cf. Claim 1(i) in KM89 p.1253)[8] and restrict attention to nonnegative prices only.

In my opinion, it is best to think of KM89's version of T83's economy with affine supply functions as the modified strategic game

$$\Gamma^{KM}\langle\mu\rangle = \left\langle \Sigma_i, U_i^{KM} \right\rangle_{i\in\{1,\ldots,N\}}$$

such that the modified—brute force—utility function is given as

$$U_i^{KM}(\sigma_i, \sigma_{-i}) = \begin{cases} -\infty & \text{if } \mu(\mathbf{p}[\sigma] < 0) > 0, \\ -\infty & \text{if } \mu(S_i[\sigma_i](\mathbf{p}[\sigma]) < 0) > 0, \\ \bar{\pi}_i(\sigma_i, \sigma_{-i}) & \text{else.} \end{cases}$$

In contrast to T83, a player is here punished through an infinite utility whenever there exists a positive probability that markets clear at a negative price or that negative amounts are supplied. If the probability measure satisfies

$$\mu(\boldsymbol{\varepsilon} < \varepsilon) > 0$$

for all $\varepsilon > 0$, any symmetric strategic profile σ with $a \neq 0$ would result in an infinite negative utility: if $a < 0$, there is some strictly positive probability of a negative price because of

$$\mu(\mathbf{p}[\sigma] < 0) = \mu\left(\frac{Na + \boldsymbol{\varepsilon}}{N\beta + m} < 0\right) = \mu(\boldsymbol{\varepsilon} < -Na) > 0;$$

if $a > 0$, there is some strictly positive probability of a negative supply because of

$$\mu(S_i[\sigma_i](\mathbf{p}[\sigma]) < 0) = \mu\left(\boldsymbol{\varepsilon} - m\frac{Na + \boldsymbol{\varepsilon}}{N\beta + m} < 0\right) = \mu\left(\boldsymbol{\varepsilon} < \frac{ma}{\beta}\right) > 0.$$

Because KM89 assume full support of $\boldsymbol{\varepsilon}$ on $[0, \infty)$, any existing symmetric affine Nash equilibria of Γ that do not start at the origin would thus be excluded in an ad hoc, i.e., brute force, fashion.

The interesting fact is that such ad hoc exclusion is not necessary. As this paper's corrected analysis of T83 shows, even without any ad hoc/brute force exclusion of negative prices and supplied amounts only positive prices and supplied amounts are possible in a symmetric Nash equilibrium of Γ with demand uncertainty. KM89 make the following statement about T83:

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[8] In KM89's own words: "We are ignoring the possibility of negative outputs." (p.1272).

> “Turnbull (1983), generalizing a result of Robson (1981), has shown for the market of this example that the function $S(p) = \frac{v_2}{w_2}p$ [with $\frac{v_2}{w_2} = \beta = \frac{1}{2}\left(-m + \sqrt{m^2 + \frac{4m}{c}}\right)$; the author] is the unique SFE when firms are restricted to choosing *linear* supply functions; with the restriction to linear supply functions, the uniqueness result is independent of the support of $\boldsymbol{\varepsilon}$ (as long as it is nondegenerate).” (p.1261)

In the light of this paper’s corrected analysis of T83 this statement can be generalized as follows: the unique symmetric Nash equilibria of the linear scenario Γ^{KM} in KM89 are also the unique symmetric Nash equilibria of the affine scenario Γ—in which firms/traders can also choose nonlinear affine supply functions—as long as the support of $\boldsymbol{\varepsilon}$ is nondegenerate under μ.

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