Strategic market games with interim price information

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Abstract

Strategic market games with interim price information combine the rational expectations concept from competitive equilibrium models under asymmetric information with Bayesian market games (Kyle 1989; Vives 2011). In this novel class of simultaneous moves games, the informed traders submit their linear demand-schedules to a Walrasian auctioneer after they have observed their private signals as well as their interim price information. Because this price information depends on the traders' actions—which are, in turn, conditional on this information—consistency and measurability issues arise that are absent for Bayesian games. As a generalization of Bayesian Nash equilibria for Bayesian market games, I show that Nash equilibria of strategic market games with interim price information may support price-collusion between the informed traders against a liquidity trader.

Keywords: Rational expectations; Bayesian Nash equilibria; Measurability; Price-collusion; Procurement

JEL Classification Numbers: C72; D82; G12; H57

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1 Introduction

I consider an asset trade economy under asymmetric information in which informed traders receive private signals about a risky asset's (common or private) value and there exists (noisy or noise-free) price-inelastic demand by some liquidity trader. In competitive rational expectations equilibrium (=REE) models (i) the informed traders know the REE price function and (ii) they condition their demand decisions on observable interim price information. In Bayesian market games the privately informed traders submit demand-schedules to a Walrasian auctioneer through which they commit themselves to buy/sell the corresponding amount of the asset at the market price that will be ex post announced by the Walrasian auctioneer. The traders thereby (i) know the Bayesian Nash equilibrium (=BNE) price function but (ii) they do not condition the choice of their demand-schedules on any interim price information.

This paper combines the rational expectations concept of observable interim price information with the concept of strategic competition over linear demand-schedules from Bayesian market games. To this purpose I construct a class of games—called *strategic market games with interim price information*—in which a trader who has observed his private signal additionally observes interim price information before he submits a linear demand-schedule to the Walrasian auctioneer. In contrast to Bayesian market games, the strategic choice of a demand-schedule is thus not only conditioned on the trader's private signal but also on his interim price information.

To be precise about this difference, consider the actions and strategies for a Bayesian market game with linear demand-schedules, denoted Γ^B, whereby I stay closely to the terminology and notation used in the seminal article by Kyle (1989). An informed trader $i \in \{1, ..., N\}$ who has observed his private signal $\theta_i \in \Theta_i \subseteq \mathbb{R}$ submits to the Walrasian auctioneer a demand-schedule

$$\left(x_i\left[\theta_i\right](p), p\right)_{p \in \mathbb{R}} \tag{1}$$

with linear demand function

$$x_i\left[\theta_i\right](p) = \mu\left[\theta_i\right] + \beta\left[\theta_i\right]\theta_i - \gamma\left[\theta_i\right]p \tag{2}$$

such that (1) commits trader i to buy/sell the respective amount $x_i\left[\theta_i\right](p)$ whenever the Walrasian auctioneer announces the market price p. Because I restrict attention to linear demand-schedules, the action of each agent $\theta_i \in \Theta_i$ effectively amounts to the choice of admissible values for the three functional parameters

$$(\mu[\theta_i], \beta[\theta_i], \gamma[\theta_i]) \in A \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}.$$

A strategy of trader i in the Bayesian market game Γ^B assigns to all his agents θ_i some action in A pinning down the parameter values for the linear demand function (2). Although the trader's demand (2) in a Bayesian market game is a function in prices $p \in \mathbb{R}$, these prices are not any interim but only ex post information after the trader's agent θ_i has chosen his demand-schedule. In a Bayesian market game, the strategic choice of a demand-schedule is thus exclusively conditioned on the information given by a trader's private signal θ_i but not on any interim price information.

Turn now to the novel concept of a strategic market game with interim price information, denoted Γ^P. Instead of (1), I consider linear demand-schedules

$$(x_i[\theta_i, \hat{p}_i, \alpha_i](p), p)_{p \in \mathbb{R}} \tag{3}$$

with

$$x_i[\theta_i, \hat{p}_i, \alpha_i](p) = \mu[\theta_i, \hat{p}_i, \alpha_i] + \beta[\theta_i, \hat{p}_i, \alpha_i]\theta_i - \gamma[\theta_i, \hat{p}_i, \alpha_i]p$$

such that $(\hat{p}_i, \alpha_i) \in \mathbb{R} \times \mathbb{R}_{>0}$ stands for interim price information that reveals to trader i the intercept $\hat{p}_i$ and the slope α_i of his linear residual demand function. The possible actions of agent $[\theta_i, \hat{p}_i, \alpha_i]$ are the same as for the Bayesian market game, i.e., all admissible parameter values

$$\mu[\theta_i, \hat{p}_i, \alpha_i], \beta[\theta_i, \hat{p}_i, \alpha_i], \gamma[\theta_i, \hat{p}_i, \alpha_i] \in A.$$

A strategy of trader i assigns now to all his agents $[\theta_i, \hat{p}_i, \alpha_i]$—and not just to all θ_i—some action that commits trader i to buy/sell in accordance with the demand-schedule (3) submitted to the Walrasian auctioneer.

In contrast to a Bayesian market game, a strategic market game with interim price information comes with the same "chicken-egg" problem that is shared by all rational expectations models: What comes first, the information-based decision or the decision-based information? Fully revealing competitive REE models typically sideline this "chicken-egg" problem by stipulating an omniscient Walrasian auctioneer who offers to all traders a REE price function that is measurable with respect to the traders' aggregate information. But such omniscient Walrasian auctioneer does not 'explain' anything as (i) it does not resemble any existing market institution whereby (ii) a fully-revealing REE price function might not be incentive compatible.[1] My preferred interpretation is that the time-sequence of information-based decisions versus decision-based information happens for rational expectations models in some unexplained "black box". One can then hope that this rational expectations "black box" works as a viable modeling shortcut for some more complex—but unmodeled—sequential situation in which traders observe information, adjust their decisions, observe new information, newly adjust their decisions, ..., and so forth.[2] Although this 'answer' to the "chicken-egg" problem of interim price information is not fully satisfactory, the present paper does not go beyond it but accepts the rational expectations concept of interim price information as a viable modeling short-cut for relevant real-life situations.

The "chicken-egg" problem of rational expectations causes a consistency as well as a measurability problem for the construction of a strategic market game with interim price information Γ^P that is absent for a Bayesian market game Γ^B. In a Bayesian market game trader i's agent who submits a demand-schedule is unambiguously pinned down by a move of nature; that is, if the true state of the world happens to be ω, then agent $\boldsymbol{\theta}_i(\omega)$—corresponding to i's private signal—submits the demand-schedule. Because all traders' interim price information has to be consistent with all traders' strategy choices in order to ensure the same market price for all traders, the situation is significantly more complex for a strategic market game with interim price information. I will show that the agents of all traders who submit in state ω their demand-schedules have to be simultaneously pinned down as some N-tuple

$$([\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma](\omega), \boldsymbol{\alpha}_i[\sigma](\omega)])_{i=1,..,N}$$

[1] To stipulate that all traders' accept a fully revealing price function might ignore the traders' incentives to hide their private information through the manipulation of prices. For example, why should the owner of a bad car in Akerlof's (1970) lemon market reveal the low quality of his car by accepting a low price? Hellwig (1980), basically, argues that rational expectations equilibria are only incentive compatible for large markets in which each atomless trader has no impact on the equilibrium price. For a game-theoretic analysis of the incentive compatibility of REE, I refer the reader to Glycopantis and Yannelis (2005).

[2] For recent modeling approaches which aim to shed some light on this "black box", see Pesce, Urbinati, and Yannelis (2024) and references therein.

whereby the traders' interim price information $\hat{\mathbf{p}}_i [\sigma] (\omega), \boldsymbol{\alpha}_i [\sigma] (\omega)$ depends—in addition to the true state ω—also on their chosen strategy profile σ.

Formally, the question whether all traders' strategy choices are consistent with their interim price information or not, is equivalent to the question of whether there exists a solution of a system of equations over all traders' interim price information in a given state of the world under a given strategy profile. The solutions—if any—for this system of equations result in an action-profile correspondence induced by a given strategy profile σ. It turns out that this correspondence induced by σ may or may not allow for the selection of a measurable function for which the traders' expected utility from strategy profile σ is well-defined.[3] As a consequence, there exist for Γ^P well-behaved versus non-well-behaved strategy-profiles such that the traders' expected utility only exists for well-behaved strategy-profiles. This is not a problem for the Bayesian market game because all strategy-profiles are, by construction, measurable functions for which each traders' expected utility is well-defined.

Due to the resolution of the "chicken-egg" problem by a consistency condition, it can be possible for a trader to influence his own residual demand function and thereby his interim price information through his strategy choice. Again, this is not possible in a Bayesian market game for which a trader's residual demand function is exclusively determined by the strategy choice of his opponents. Therefore, a strategic market game with interim price information is not a Bayesian game and the Nash equilibria of Γ^P can, in general, not be interpreted as Bayesian Nash equilibria. My analysis of Nash equilibria for strategic market games with interim price information comes with two main insights.

1. If a BNE of Γ^B satisfies a specific sufficiency condition—according to which interim price information is irrelevant to a best response—, the corresponding BNE outcome can be carried over to a Nash equilibrium of Γ^P. In particular, this sufficiency condition is satisfied by the constant, symmetric BNE for the Bayesian market games considered in Kyle (1989) and Vives (2011).

[3] Im grateful to Ulrich Horst for pointing me to the Kuratowski–Ryll-Nardzewski Selection Theorem in Aliprantis and Border (2006).

2. In contrast to the BNE of Γ^B, there may exist Nash equilibria of Γ^P which support price-collusion between the informed traders against the liquidity trader. The intuitive reason for this economically relevant finding is straightforward: if the traders can condition their submitted demand-schedules on interim price information, they can design demand-schedules that punish any strategies which deviate from the 'agreed-upon' price.

The conceptually novel part of this second main insight is that the triggering of punishing actions happens within the simultaneous move game Γ^P for which no time-sequence of the traders' moves is defined.

Relation to Kyle (1989) and Vives (2011)

The idea to use a trader's residual demand function as his interim price information is inspired by the seminal articles of Kyle (1989) and Vives (2011). To establish the existence of constant, symmetric BNE[4], Kyle (1989) and Vives (2011) use the following "irrelevance of interim price information" argument:

> Suppose that every agent could condition his action on the parameters of his residual demand function. If his optimal response is always the same action irrespective of this interim price information, this action must also be a best response whenever the agent does not observe any interim price information.

Importantly, neither Kyle (1989) nor Vives (2011) go beyond Bayesian market games as their traders cannot actually observe any interim price information.

A major motivation for the models of Kyle (1989) and Vives (2011) is to compare the *informativeness* of their respective BNE price functions with the informativeness of the REE price functions from competitive REE models. The discussion of the informativeness of equilibrium

[4] Kyle (1989) establishes existence of constant, symmetric BNE within a CARA-Gaussian, common value model. Existence of such BNE fails in Kyle's model if the demand of the liquidity trader is noise-free. Vives (2011) establishes existence of constant, symmetric BNE within a risk–neutral, Gaussian framework for noise-free liquidity demand and a quadratic cost function. In Vives's model existence of such BNE fails if the private asset value model becomes a common value model.

price functions relates to the question which theoretical models may support Fama's (1979) Efficient Market Hypothesis. In competitive REE models the informed traders condition their demand decisions on interim information revealed through the REE price function. Radner (1979)—for the case of a finite state space—and Allen (1981)—for the case of a general probability space—show that the information revealed by the REE market price function is for noise-free liquidity demand generically one-to-one to the full communication information, i.e., to the information that would obtain if all traders truthfully shared their private information. Early examples of "fulfilled expectations" (=rational expectations) equilibria appear in Green (1975), Grossman (1976, 1977, 1978), and Kreps (1977). Although *noisy* REE models with normally distributed liquidity demand keep the REE market price function from fully revealing, the market price function of noisy REE converges to fully revealing prices if the variance of the noisy liquidity demand goes towards zero (cf. Hellwig 1980; Grossman and Stiglitz 1980).

The BNE price function in Kyle's (1989) common value model remains bounded away from being fully revealing if the variance of the noisy liquidity demand converges to zero. In contrast, the BNE price function is fully revealing for Vives's (2011) noise-free, not common value model. However, whether the BNE price function reveals any private information or not, is completely irrelevant to the traders' strategic choice in Kyle's and Vives's models as their respective BNE are derived through the above "irrelevance of interim price information" argument. This raises the following question: If the traders themselves do not care about the informativeness of the BNE price function when choosing their best responses, why should we—as external observers—care about the price informativeness of these BNE? Put differently, in the constant, symmetric BNE of Kyle (1989) and Vives (2011) it does not matter to the actual market participants' decisions whether the markets are information-efficient in Fama's sense or not.

Strategic market games with interim price information can generate different economic outcomes than Bayesian market games (e.g., price-collusion) exactly because the observation of interim price information may be relevant to the traders' strategic choices. The relevant information that is revealed by prices in a Nash equilibrium of a strategic market game with interim price information does not necessarily concern the 'true' asset value—as in competitive REE models—but rather the strategy choices of the other traders. For example, the price-collusion Nash equilibrium that I derive for the procurement scenario does not reveal any trader's private information. Interim price information only assists the traders to coordinate towards price-collusion against the liquidity trader because deviating behavior can be punished based on this price information. A highly *informative* price function would thus be rather bad news for the government in a procurement situation because it might support price-collusive behavior.

The remainder of this paper proceeds as follows. Section 2 sets up the probability space and recalls relevant concepts from measure theory. Bayesian market games are constructed in Section 3. Section 4 presents a sufficiency condition for identifying BNE. Strategic market games with interim price information are constructed in Section 5. In Section 6, I demonstrate the possibility of price-collusion in a procurement situation. Section 7 concludes. All propositions are proved in the Appendix. The Supplemental Appendix revisits—and slightly extends—Kyle's (1989) original analysis of constant, symmetric BNE for Bayesian market games within his CARA-Gaussian framework.

2 Preliminaries

2.1 The probability space

Let $V_i \subseteq \mathbb{R}$ denote the set of possible asset values from the perspective of (informed) trader $i \in \{1, ..., N\}$. Denote by $\Theta_i \subseteq \mathbb{R}$ the signal space of $i \in \{1, ..., N\}$. Denote by $Z \subseteq \mathbb{R}$ the set of possible values for the inelastic demand of the liquidity trader. Define the probability space $(\Omega, \mathcal{B}, \pi)$ such that the state space is given as

$$\Omega = V_1 \times \cdots \times V_N \times \Theta_1 \times \cdots \times \Theta_N \times Z \subseteq \mathbb{R}^{2N+1},$$

with $\omega = (v_1, ..., v_N, \theta_1, ..., \theta_N, z) \in \Omega$ as generic element; $\mathcal{B}$ denotes the Borel sigma algebra generated by the Euclidean product topology on Ω; the probability measure π stands for the traders' common prior.

For any $X \subseteq \mathbb{R}^M$ I write $\mathcal{B}(X)$ for the Borel sigma algebra generated by the Euclidean product topology on X. Recall that a function $f : (\Omega, \mathcal{B}) \to (X, \mathcal{B}(X))$, or simply written $f : \Omega \to X$, is $\mathcal{B}$-measurable iff

$$A \in \mathcal{B}(X) \text{ implies } f^{-1}(A) \in \mathcal{B}$$

with the pre-image of A under f being defined as

$$f^{-1}(A) = \{\omega \in \Omega \mid f(\omega) \in A\}.$$

If f is measurable with respect to any subalgebra $\mathcal{G} \subseteq \mathcal{B}$, it is also $\mathcal{B}$-measurable. Random variables are $\mathcal{B}$-measurable functions from $(\Omega, \mathcal{B})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

I will use the following coordinate random variables

$$\begin{aligned}
\mathbf{v}_i(v_1, ..., v_N, \theta_1, ..., \theta_N, z) &= v_i,\ i \in \{1, ..., N\},\\
\boldsymbol{\theta}_i(v_1, ..., v_N, \theta_1, ..., \theta_N, z) &= \theta_i,\ i \in \{1, ..., N\},\\
\mathbf{z}(v_1, ..., v_N, \theta_1, ..., \theta_N, z) &= z,
\end{aligned}$$

which stand, respectively, for trader i's private asset value, trader i's private signal, and the demand by the liquidity trader. I speak of a *common value* model iff there exists some random variable $\mathbf{v}$ such that, for all i, $\mathbf{v}_i(\omega) = \mathbf{v}(\omega)$ for π-almost all ω. The demand of the liquidity trader is *noise-free* iff

$$\pi(\mathbf{z} = z) = 1 \text{ for some } z \in Z$$

and *noisy* else.

2.2 Conditional expectations

A random variable $\mathbf{u} : \Omega \to \mathbb{R}$ is (Lebesgue) *integrable* wrt π iff[5]

$$\int_\Omega \mathbf{u}^+ d\pi < \infty \text{ and } \int_\Omega \mathbf{u}^- d\pi < \infty$$

with

$$\begin{aligned}
\mathbf{u}^+(\omega) &= \begin{cases} \mathbf{u}(\omega) & \text{if } \mathbf{u}(\omega) \geq 0 \\ 0 & \text{if } \mathbf{u}(\omega) \leq 0 \end{cases},\\
\mathbf{u}^-(\omega) &= \begin{cases} \mathbf{0} & \text{if } \mathbf{u}(\omega) \geq 0 \\ -\mathbf{u}(\omega) & \text{if } \mathbf{u}(\omega) \leq 0. \end{cases}
\end{aligned}$$

[5] The definitions and notations follow closely Section 34 in Billingsley (1996).

Consider any sigma-algebra $\mathcal{G} \subseteq \mathcal{B}$. If a random variable $\mathbf{u}$ is integrable, then there exists, by the Radon-Nikodym Theorem, a random variable

$$E\left(\mathbf{u} \,\|\mathcal{G}\right) : \Omega \rightarrow \mathbb{R} \tag{4}$$

such that

1. $E\left(\mathbf{u} \,\|\mathcal{G}\right)$ is $\mathcal{G}$-measurable and integrable;

2. $E\left(\mathbf{u} \,\|\mathcal{G}\right)$ satisfies for all $G \in \mathcal{G}$ the functional equation
$$\int_G E\left(\mathbf{u} \,\|\mathcal{G}\right) d\pi = \int_G \mathbf{u} d\pi. \tag{5}$$

Definition. *Any version of* $E\left(\mathbf{u} \,\|\mathcal{G}\right)$ *is called the conditional expected value of* $\mathbf{u}$ *wrt* $\mathcal{G}$.

Because $\Omega \in \mathcal{G}$ for any $\mathcal{G} \subseteq \mathcal{B}$, (5) yields for $G = \Omega$ the *law of iterated expectations*:

$$\begin{aligned} \int_\Omega E\left(\mathbf{u} \,\|\mathcal{G}\right) d\pi &= \int_\Omega \mathbf{u} d\pi \\ &\Leftrightarrow \\ E\left(E\left(\mathbf{u} \,\|\mathcal{G}\right)\right) &= E\left(\mathbf{u}\right). \end{aligned} \tag{6}$$

2.3 Sigma-algebras generated by random variables

Consider a collection of random variables $\mathbf{y}_1, ..., \mathbf{y}_m$. I denote by $\mathcal{G}\left(\mathbf{y}_1, ..., \mathbf{y}_m\right) \subseteq \mathcal{B}$ the sigma-algebra generated by $\mathbf{y}_1, ..., \mathbf{y}_m$, which is the smallest sigma-algebra for which every $\mathbf{y}_k$, $k = 1, ..., m$, becomes measurable. Note that we have

$$\int_\Omega E\left(\mathbf{u} \,\|\mathcal{G}\left(\mathbf{y}_1, ..., \mathbf{y}_m\right)\right) d\pi = \int_\Omega \mathbf{u} d\pi\left(\cdot \mid \mathbf{y}_1, ..., \mathbf{y}_m\right)$$

with $\pi\left(\cdot \mid \mathbf{y}_1, ..., \mathbf{y}_m\right)$ denoting the corresponding conditional probability measure. For a single random variable $\mathbf{y}$ we have the explicit expression

$$\mathcal{G}\left(\mathbf{y}\right) = \left\{\mathbf{y}^{-1}\left(B\right) \text{ for all } B \in \mathcal{B}\left(X\right)\right\}.$$

Consider, for example, the familiar case such that the private signal random variable $\boldsymbol{\theta}_i$ has full support on the finite set $\Theta_i = \{\theta_1, ..., \theta_m\}$. Then $\mathcal{G}\left(\boldsymbol{\theta}_i\right)$ is generated by the partition

$$\Omega_1, ..., \Omega_m$$

such that

$$\Omega_k = \{\omega \in \Omega \mid \boldsymbol{\theta}_i(\omega) = \theta_k\} = (\boldsymbol{\theta}_i = \theta_k).$$

In that case we obtain as conditional expected value of $\mathbf{u}$ in any state ω with $\boldsymbol{\theta}_i(\omega) = \theta_k$

$$E(\mathbf{u} \,\|\mathcal{G}(\boldsymbol{\theta}_i))(\omega) = \frac{1}{\pi(\boldsymbol{\theta}_i = \theta_k)} \int_{\Omega_k} \mathbf{u} d\pi = \int_{\Omega_k} \mathbf{u} d\pi(\cdot \mid \boldsymbol{\theta}_i = \theta_k)$$

with conditional probability measure

$$\pi(B \mid \boldsymbol{\theta}_i = \theta_k) = \frac{\pi(B \cap (\boldsymbol{\theta}_i = \theta_k))}{\pi(\boldsymbol{\theta}_i = \theta_k)} \text{ for all } B \in \mathcal{B}.$$

Two collections of random variables $\mathbf{y}_1, ..., \mathbf{y}_m$ and $\mathbf{y}'_1, ..., \mathbf{y}'_n$ are *informationally equivalent* iff

$$\mathcal{G}(\mathbf{y}_1, ..., \mathbf{y}_m) = \mathcal{G}(\mathbf{y}'_1, ..., \mathbf{y}'_n).$$

Informational equivalence holds iff there is a one-to-one mapping f such that

$$(\mathbf{y}_1, ..., \mathbf{y}_m) = f(\mathbf{y}'_1, ..., \mathbf{y}'_n) \text{ for all } \omega.$$

3 Bayesian market games

In a Bayesian market game the informed traders simultaneously submit their linear demand-schedules to the Walrasian auctioneer after they have observed their private signals but before they receive any price information.

3.1 Strategies

The set of actions

$$A \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$$

contains the admissible values of the three different functional parameters that enter into a linear demand-schedule submitted to the Walrasian auctioneer.

Definitions. *The set of strategies of trader i, denoted Σ_i^B, consists of all $\mathcal{G}(\boldsymbol{\theta}_i)$-measurable functions $\sigma_i : \Omega \to A$ such that*

$$\sigma_i(\omega) = (\mu[\boldsymbol{\theta}_i(\omega)], \beta[\boldsymbol{\theta}_i(\omega)], \gamma[\boldsymbol{\theta}_i(\omega)]).$$

The set of strategy-profiles of Γ^B is given as $\Sigma^B = \times_{i=1}^N \Sigma_i^B$ with generic element $\sigma : \Omega \to A^N$.

Given strategy σ_i and $\omega \in \Omega$, the resulting action

$$\sigma_i(\omega) = (\mu[\boldsymbol{\theta}_i(\omega)], \beta[\boldsymbol{\theta}_i(\omega)], \gamma[\boldsymbol{\theta}_i(\omega)]) \in A$$

induces the demand function

$$x_i[\boldsymbol{\theta}_i(\omega)](p) = \mu[\boldsymbol{\theta}_i(\omega)] + \beta[\boldsymbol{\theta}_i(\omega)] - \gamma[\boldsymbol{\theta}_i(\omega)]\, p \tag{7}$$

which is linear in p with slope $-\gamma[\boldsymbol{\theta}_i(\omega)] < 0$ and intercept $\mu[\boldsymbol{\theta}_i(\omega)] + \beta[\boldsymbol{\theta}_i(\omega)] \in \mathbb{R}$. The economic interpretation of action $\sigma_i(\omega)$ is as follows. After trader i receives in state ω his private signal $\boldsymbol{\theta}_i(\omega)$, he submits to the Walrasian auctioneer the demand-schedule

$$(x_i[\boldsymbol{\theta}_i(\omega)](p), p)_{p \in \mathbb{R}}$$

through which he commits himself to buy (resp. sell) amount (7) at price p if the auctioneer announces p as market price. In what follows I also refer to $\boldsymbol{\theta}_i(\omega)$ as trader i's agent in state ω.

I call $\sigma_i \in \Sigma_i^B$ a *constant* strategy iff for all ω

$$\sigma_i(\omega) = (\mu_i, \beta_i, \gamma_i) \in A.$$

Note that a constant strategy does not mean that the trader submits the same demand-schedule in every state of the world. A constant strategy σ_i would induce a constant demand-schedule if it additionally satisfies $\beta_i = 0$ so that for all ω

$$x_i[\boldsymbol{\theta}_i(\omega)](p) = \mu_i - \gamma_i p.$$

3.2 Market price function

A price p clears markets in state $(\cdot, \theta_1, ...\theta_N, z) \in \Omega$ under strategy profile $\sigma \in \Sigma^B$ iff

$$\sum_{i=1}^N x_i[\theta_i](p) + z = 0 \Leftrightarrow p = \frac{\sum_{i=1}^N (\mu[\theta_i] + \beta[\theta_i]\theta_i) + z}{\sum_{i=1}^N \gamma[\theta_i]}.$$

By our assumptions $\gamma[\cdot] > 0$ and $p \in \mathbb{R}$, we have for every $\omega \in \Omega$ under every $\sigma \in \Sigma^B$ a unique market price that will be announced by the Walrasian auctioneer.

Definition. Market price function for the Bayesian market game. *The market price function* $\mathbf{p}[\sigma] : \Omega \to \mathbb{R}$ *under strategy profile* $\sigma \in \Sigma^B$ *is given as*

$$\mathbf{p}[\sigma](\omega) = \frac{\sum_{i=1}^{N} (\mu[\boldsymbol{\theta}_i(\omega)] + \beta[\boldsymbol{\theta}_i(\omega)]\,\boldsymbol{\theta}_i(\omega)) + \mathbf{z}(\omega)}{\sum_{i=1}^{N} \gamma[\boldsymbol{\theta}_i(\omega)]}. \tag{8}$$

A word about measurability. Since all strategies $\sigma_i \in \Sigma_i^B$, $i \in \{1, ..., N\}$, are, by assumption, $\mathcal{G}(\boldsymbol{\theta}_i)$-measurable, all strategy profiles $\sigma \in \Sigma^B$ are $\mathcal{B}$-measurable (cf. 4.49 Lemma in Aliprantis and Border 2006). The market price function $\mathbf{p}[\sigma] : \Omega \to \mathbb{R}$ under σ is formally a composition $(\mathbf{p} \circ (\sigma, \mathbf{z})) : \Omega \to \mathbb{R}$ of the functions $\mathbf{p} : (A^N \times \mathbb{R}, \mathcal{B}(A^N \times \mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\sigma, \mathbf{z}) : (\Omega, \mathcal{B}) \to (A^N \times \mathbb{R}, \mathcal{B}(A^N \times \mathbb{R}))$. With $\mathbf{p}$ being continuous in $(\mu, \beta, \gamma, z) \in A^N \times \mathbb{R}$ (and therefore $\mathcal{B}(A^N \times \mathbb{R})$-measurable) and $(\sigma, \mathbf{z})$ being $\mathcal{B}$-measurable, the market price function (8) is, for every $\sigma \in \Sigma^B$, $\mathcal{B}$-measurable (cf. 4.22 Lemma in Aliprantis and Border 2006).

3.3 Utility and Bayesian Nash equilibria

All traders are expected utility maximizers who have the same strictly increasing Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ defined over payoffs. I distinguish between two different plausible economic scenarios, i.e., the market versus the procurement scenario. Loosely speaking, the market scenario corresponds to a double-auction situation in which all traders (informed and liquidity) could act as buyers and sellers. In contrast, the procurement scenario resembles a single-auction situation as I will take away any incentives for an informed trader to compete as a potential buyer with the liquidity trader who has always positive demand for the service/product, i.e., $\pi(\mathbf{z} > 0) = 1$.

Market scenario. The trader i's $\mathcal{B}$-measurable payoff function $\mathbf{g}_i[\sigma] : \Omega \to \mathbb{R}$ satisfies

$$\mathbf{g}_i[\sigma] = (\mathbf{v}_i - \mathbf{p}[\sigma])\, x_i[\boldsymbol{\theta}_i](\mathbf{p}[\sigma](\omega)) + t(x_i[\boldsymbol{\theta}_i](\mathbf{p}[\sigma])) \tag{9}$$

whereby

$$(\mathbf{v}_i - \mathbf{p}[\sigma])\, x_i[\boldsymbol{\theta}_i](\mathbf{p}[\sigma])$$

stands for gains-from-trade which hold for arbitrary (negative or positive) asset positions. The function $t : \mathbb{R} \to \mathbb{R}$, giving rise to a $\mathcal{B}$-measurable composition $t \circ (x_i[\boldsymbol{\theta}_i](\cdot)) : \Omega \to \mathbb{R}$, covers any additional cost or technological benefits that exclusively depend on the traded amount. In Kyle's

(1989) model we simply have $t(x) = 0$, $x \in \mathbb{R}$; Vives (2011) considers $t(x) = -\frac{\lambda}{2}x^2$, $x \in \mathbb{R}$, with parameter $\lambda \geq 0$.

Procurement scenario. In the procurement scenario the liquidity trader (e.g., the government) acts as a monopsonist who has the institutional power to avoid any positive demand competition from informed traders. Formally, I define trader i's $\mathcal{B}$-measurable payoff function $\mathbf{g}_i[\sigma] : \Omega \to \mathbb{R}$ under σ as (9) whenever $x_i[\boldsymbol{\theta}_i](\mathbf{p}[\sigma]) < 0$ but as $\mathbf{g}_i[\sigma] = 0$ else. That is, any trader who demands a non-negative amount at the market price will now be punished with a zero payoff. To keep the model simple, I assume that the remaining traders will sell their aggregate supply to the liquidity trader at the original market price $\mathbf{p}[\sigma](\omega)$.

Bernoulli utility. Write $\mathbf{u}_i[\sigma]$ for the composition $u \circ \mathbf{g}_i[\sigma]$ such that, for all $\omega \in \Omega$,

$$(u \circ \mathbf{g}_i[\sigma])(\omega) = u(\mathbf{g}_i[\sigma](\omega))$$

Since the Bernoulli utility function $u(\cdot)$ is—as an increasing function—$\mathcal{B}(\mathbb{R})$-measurable and $\mathbf{g}_i[\sigma]$ is $\mathcal{B}$-measurable for either economic scenario, the utility random variable $\mathbf{u}_i[\sigma] : \Omega \to \mathbb{R}$ is $\mathcal{B}$-measurable. I further assume that $\mathbf{u}_i[\sigma]$ is integrable wrt π for every $\sigma \in \Sigma^B$.[6]

Definition. *The utility of trader i from strategy-profile $\sigma \in \Sigma^B$ is given as his expected utility*

$$U_i^B(\sigma) = E(\mathbf{u}_i[\sigma]) = \int_\Omega \mathbf{u}_i[\sigma]\, d\pi.$$

Collecting the above components yields the Bayesian market game

$$\Gamma^B = \left\langle (\Omega, \mathcal{B}, \pi), \Sigma_i^B, U_i^B \right\rangle_{i \in \{1,\dots,N\}}. \tag{10}$$

Definition. Bayesian Nash equilibria (=BNE). *The strategy profile*

$$(\sigma_1^*, \dots, \sigma_N^*) = \left((\sigma_1^*[\theta_1])_{\theta_1 \in \Theta_1}, \dots, (\sigma_N^*[\theta_N])_{\theta_N \in \Theta_N} \right) \in \Sigma^B \tag{11}$$

[6]Technically speaking, I only allow for Bernoulli utility functions which imply $\mathbf{u}_i[\sigma] \in L_1(\pi)$ for all $\sigma \in \Sigma^B$ (cf. Chapter 13 on L_p-spaces in Aliprantis and Border 2006). Although this depends, in general, on the measure π, one could simply ensure integrability of $\mathbf{u}_i[\sigma]$ for all $\sigma \in \Sigma^B$ wrt any probability measure π by restricting attention to $u(\cdot)$ which are bounded from above and below implying, for all $\sigma \in \Sigma^B$, $\mathbf{u}_i[\sigma] \in L_\infty(\pi) \subseteq L_1(\pi)$ for all π.

is a BNE of Γ^B *iff*

$$U_i^B\left(\sigma_i^*, \sigma_{-i}^*\right) \geq U_i^B\left(\sigma_i, \sigma_{-i}^*\right) \text{ for all } \sigma_i \in \Sigma_i^B$$

for every $i \in \{1, ..., N\}$*.*

4 A sufficiency condition for identifying Bayesian Nash equilibria

This section develops in detail the "irrelevance of interim price information" argument that has been originally employed by Kyle (1989) and Vives (2011) for the derivation of their constant, symmetric BNE. Importantly, any BNE which can be established through this "irrelevance of interim price information" argument will carry over to 'become' Nash equilibria for strategic market games with interim price information.

4.1 The (unobserved) residual demand function

The market price function (8) can be equivalently characterized in terms of the *residual demand function* of any trader i through the equation

$$\mathbf{p}[\sigma] = \hat{\mathbf{p}}_i[\sigma_{-i}] + \boldsymbol{\alpha}_i[\sigma_{-i}]\left(\mu[\boldsymbol{\theta}_i] + \beta[\boldsymbol{\theta}_i]\boldsymbol{\theta}_i - \gamma[\boldsymbol{\theta}_i]\mathbf{p}[\sigma]\right)$$

such that the intercept and the slope of i's linear residual demand function are, respectively, given as the random variable $\hat{\mathbf{p}}_i[\sigma_{-i}] : \Omega \to \mathbb{R}$ with

$$\hat{\mathbf{p}}_i[\sigma_{-i}] = \frac{\sum_{j \neq i}\left(\mu[\boldsymbol{\theta}_j] + \beta[\boldsymbol{\theta}_j]\boldsymbol{\theta}_j\right) + \mathbf{z}}{\sum_{j \neq i}\gamma[\boldsymbol{\theta}_j]}$$

and as the random variable $\boldsymbol{\alpha}_i[\sigma_{-i}] : \Omega \to \mathbb{R}$ with

$$\boldsymbol{\alpha}_i[\sigma_{-i}] = \frac{1}{\sum_{j \neq i}\gamma[\boldsymbol{\theta}_j]}.$$

Observe that for all $\sigma \in \Sigma^B$

$$\mathcal{G}(\boldsymbol{\theta}_i, \mathbf{p}[\sigma]) \subseteq \mathcal{G}(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma_{-i}], \boldsymbol{\alpha}_i[\sigma_{-i}]),$$

because of

$$\mathbf{p}[\sigma] = \frac{\hat{\mathbf{p}}_i[\sigma_{-i}] + \boldsymbol{\alpha}_i[\sigma_{-i}](\mu[\boldsymbol{\theta}_i] + \beta[\boldsymbol{\theta}_i]\boldsymbol{\theta}_i)}{1 + \boldsymbol{\alpha}_i[\sigma_{-i}]\gamma[\boldsymbol{\theta}_i]}.$$

We have informational equivalence

$$\mathcal{G}(\boldsymbol{\theta}_i, \mathbf{p}[\sigma]) = \mathcal{G}(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma_{-i}]) = \mathcal{G}(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma_{-i}], \boldsymbol{\alpha}_i[\sigma_{-i}]) \tag{12}$$

for any $\sigma \in \Sigma^B$ such that for all $j \neq i$

$$\gamma[\boldsymbol{\theta}_j(\omega)] = \gamma_j \text{ for all } \omega$$

as $\boldsymbol{\alpha}_i[\sigma_{-i}]$ becomes then a constant.

4.2 The "irrelevance of interim price information" argument

Although trader i does not observe in a Bayesian market game any interim price information before he submits his demand-schedule to the Walrasian auctioneer, let us entertain the thought experiment that he could condition his demand-schedule not only on his private signal $\boldsymbol{\theta}_i$ but also on the values of $\hat{\mathbf{p}}_i[\sigma_{-i}]$ and $\boldsymbol{\alpha}_i[\sigma_{-i}]$.

Proposition 1.

- *Suppose that $\sigma^* \in \Sigma^B$ satisfies the following "irrelevance of interim price information" condition: for π-almost all $\omega \in \Omega$*

$$\begin{aligned} & E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)(\omega) \\ \geq\ & E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)(\omega) \end{aligned} \tag{13}$$

  for all $\sigma_i \in \Sigma_i^B$ for every $i \in \{1, ..., N\}$.

- *Then $\sigma^* \in \Sigma^B$ is a BNE of the Bayesian market game Γ^B.*

The identification of constant, symmetric BNE in Kyle (1989) and Vives (2011) can be best thought of as a special case of Proposition 1. Focus on a market scenario in which i's opponents choose a constant, symmetric strategy profile σ_{-i}^* such that for all $j \neq i$

$$\sigma_j^*(\omega) = (\mu, \beta, \gamma) \text{ for all } \omega.$$

In that case the parameters of i's residual demand function become

$$\hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right] = \frac{(N-1)\,\mu + \beta\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\,\gamma}$$

and

$$\boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right] = \alpha_i\left[\sigma_{-i}^*\right] = \frac{1}{(N-1)\,\gamma},$$

implying the informational equivalence (12). Suppose that there exists some optimal demand of i against σ_{-i}^* as a function in $(\boldsymbol{\theta}_i, \mathbf{p})$ given as the maximizer

$$\begin{aligned} & x_i^*(\boldsymbol{\theta}_i, \mathbf{p}) && (14)\\ \in\ & \arg\max_{x_i\in\mathbb{R}} E\left(u\left(\left(\mathbf{v}_i - \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right] - \alpha_i\left[\sigma_{-i}^*\right]x_i\right)x_i + t\,(x_i)\right) \parallel \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right) \end{aligned}$$

such that

$$\mathbf{p} = \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right] + \alpha_i\left[\sigma_{-i}^*\right]x_i^*(\boldsymbol{\theta}_i, \mathbf{p})\,.$$

If one can establish that there exist parameter values

$$(\mu^*, \beta^*, \gamma^*) \in A$$

such that

$$x_i^*(\boldsymbol{\theta}_i, \mathbf{p}) = \mu^* + \beta^*\boldsymbol{\theta}_i - \gamma^*\mathbf{p}, \tag{15}$$

one has shown that

$$\sigma_i^*(\omega) = (\mu^*, \beta^*, \gamma^*) \text{ for all } \omega$$

is a best response of trader i against σ_{-i}^*. Such a constant best response strategy satisfies the "irrelevance of interim price information" condition (13) for i because the same induced demand-schedule (15) is, by construction, utility maximizing against σ_{-i}^* for all $\boldsymbol{\theta}_i(\omega)\,, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right](\omega)\,, \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right](\omega)$, $\omega \in \Omega$. By Proposition 1, the strategy profile σ^* is thus a constant, symmetric BNE.

The existence of a maximizer (14) for any given $(\boldsymbol{\theta}_i, \mathbf{p})$ is not a problematic issue under standard regularity conditions. The challenge is to identify conditions for which the linear structure (15) holds. Let me elaborate.

Example 1. Suppose that the traders are risk-neutral, i.e., $u(x) = x$, and that $t(x) = 0$. The conditional expected utility of i wrt $\mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right]\right)$ as a function in demand x_i is then given as

$$\left(E\left(\mathbf{v}_i \parallel \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}\right]\right)\right) - \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right] - \alpha_i\left[\sigma^*_{-i}\right] x_i\right) x_i.$$

The utility maximizing demand is pinned down by the FOC as

$$x^*_i = \frac{E\left(\mathbf{v}_i \parallel \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right]\right)\right) - \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right]}{2\alpha_i\left[\sigma^*_{-i}\right]}.$$

Substituting

$$\hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right] = \mathbf{p} + \alpha_i\left[\sigma^*_{-i}\right] x^*_i$$

and rearranging yields the maximizer (14) as a function in $(\boldsymbol{\theta}_i, \mathbf{p})$:

$$x^*_i\left(\boldsymbol{\theta}_i, \mathbf{p}\right) = \frac{E\left(\mathbf{v}_i \parallel \mathcal{G}\left(\boldsymbol{\theta}_i, \mathbf{p}\right)\right) - \mathbf{p}}{3\alpha_i\left[\sigma^*_{-i}\right]}.$$

The challenge for establishing a BNE in constant, symmetric strategies would be to identify

$$\sigma^*_i(\omega) = (\mu^*, \beta^*, \gamma^*) \in A \text{ for all } \omega \text{ and } i$$

such that

$$\mu^* + \beta^*\boldsymbol{\theta}_i - \gamma^*\mathbf{p}\left[\sigma^*\right] = \frac{E\left(\mathbf{v}_i \parallel \mathcal{G}\left(\boldsymbol{\theta}_i, \mathbf{p}\left[\sigma^*\right]\right)\right) - \mathbf{p}\left[\sigma^*\right]}{3\left((N-1)\gamma^*\right)^{-1}} \tag{16}$$

holds.□

Vives (2011) analyzes a risk-neutral Gaussian framework with noise-free liquidity demand. He shows that—under suitable parameter conditions—there exist a constant, symmetric BNE if one (i) deviates from the common value assumption (ii) and $t(x) = -\frac{\lambda}{2}x^2$ (cf. Proposition 1 in Vives (2011). For the reader's benefit I review and slightly extend in the Supplemental Appendix Kyle's (1989) original derivation of constant, symmetric BNE for a CARA-Gaussian, common value framework with noisy liquidity demand (Proposition 4). I also show why such derivation fails if the liquidity demand is noise-free (Proposition 5).

5 Strategic market games with interim price information

In a strategic market game with interim price information, denoted Γ^P, the informed traders simultaneously submit their linear demand-schedules to the Walrasian auctioneer after they have observed their private signals and their interim price information.

5.1 Strategies

For the Bayesian market game Γ^B strategies were defined as $\mathcal{G}(\boldsymbol{\theta}_i)$-measurable functions resulting in $\mathcal{B}$-measurable strategy-profiles. It turns out that the "chicken-egg" problem arising from the rational expectations concept of interim price information makes such a direct approach impossible for Γ^P.

5.1.1 Set-up

In contrast to a Bayesian market game, the agents of trader i are now not only characterized by their private signals $\theta_i \in \Theta_i$ but additionally by their interim price information $(\hat{p}_i, \alpha_i) \in \mathbb{R} \times \mathbb{R}_{>0}$ such that $\hat{p}_i$ denotes the intercept and α_i the slope of i's residual demand function (cf. Section 4.1). A strategy of trader i assigns to every agent $[\theta_i, \hat{p}_i, \alpha_i] \in \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0}$ some action in the set

$$A \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$$

with generic element $(\mu[\theta_i, \hat{p}_i, \alpha_i], \beta[\theta_i, \hat{p}_i, \alpha_i], \gamma[\theta_i, \hat{p}_i, \alpha_i])$.

Definitions.

(i) *The set of strategies of trader i, denoted Σ_i^P, consists of all functions $\sigma_i : \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0} \to A$.*

(ii) *The set of strategy-profiles of Γ^P is given as $\Sigma^P = \times_{i=1}^N \Sigma_i^P$ with generic element σ.*

The interpretation of the action

$$\sigma_i[\theta_i, \hat{p}_i, \alpha_i] = (\mu[\theta_i, \hat{p}_i, \alpha_i], \beta[\theta_i, \hat{p}_i, \alpha_i], \gamma[\theta_i, \hat{p}_i, \alpha_i]) \in A \tag{17}$$

chosen by agent $[\theta_i, \hat{p}_i, \alpha_i]$ is as follows. Suppose that trader i receives his private signal θ_i. When he additionally observes the interim price information $(\hat{p}_i, \alpha)_i$, he submits to the Walrasian auctioneer the demand-schedule

$$\left(x_i\left[\theta_i, \hat{p}_i, \alpha_i\right](p), p\right)_{p\in\mathbb{R}}. \tag{18}$$

Through this demand-schedule trader i commits himself to buy/sell the amount

$$x_i\left[\theta_i, \hat{p}_i, \alpha_i\right](p) = \mu\left[\theta_i, \hat{p}_i, \alpha_i\right] + \beta\left[\theta_i, \hat{p}_i, \alpha_i\right]\theta_i - \gamma\left[\theta_i, \hat{p}_i, \alpha_i\right]p \tag{19}$$

at price p whenever the auctioneer announces p as market price.

5.1.2 The "chicken-egg" problem

The conceptual challenge is to determine which demand-schedules are submitted by which agents in any given state of the world ω under strategy profile $\sigma \in \Sigma^P$. This was trivial for the Bayesian market game Γ^B because in state ω the agents $\boldsymbol{\theta}_i(\omega) = \theta_i$, $i \in \{1, ..., N\}$, submitted demand-schedules induced by the action-profile

$$\left(\mu\left[\theta_i\right], \beta\left[\theta_i\right], \gamma\left[\theta_i\right]\right)_{i\in\{1,\ldots,N\}} \in A.$$

As a consequence, I could define a strategy of trader i as a $\mathcal{G}(\boldsymbol{\theta}_i)$-measurable function into the action space A. The situation is significantly more complex for a strategic market game with interim price information Γ^P because each agent i's interim price information $(\hat{p}_i, \alpha_i)$ has to be consistent with his opponents' interim price information $(\hat{p}_j, \alpha_j)$, $j \neq i$, which in turn might depend on all agents' actions.

To illustrate this challenge, recall that the market price in a Bayesian market game was unambiguously pinned down under $\sigma \in \Sigma^B$ in every state ω as

$$\mathbf{p}\left[\sigma\right](\omega) = \hat{\mathbf{p}}_i\left[\sigma_{-i}\right](\omega) + \boldsymbol{\alpha}_i\left[\sigma_{-i}\right](\omega)\, x_i\left[\boldsymbol{\theta}_i(\omega)\right] \tag{20}$$

such that

$$\begin{aligned}
\hat{\mathbf{p}}_i\left[\sigma_{-i}\right](\omega) &= \frac{\sum_{j\neq i}\mu\left[\boldsymbol{\theta}_j(\omega)\right] + \beta\left[\boldsymbol{\theta}_j(\omega)\right]\boldsymbol{\theta}_j(\omega) + \mathbf{z}(\omega)}{\sum_{j\neq i}\gamma\left[\boldsymbol{\theta}_j(\omega)\right]}, \\
\boldsymbol{\alpha}_i\left[\sigma_{-i}\right](\omega) &= \frac{1}{\sum_{j\neq i}\gamma\left[\boldsymbol{\theta}_j(\omega)\right]}
\end{aligned}$$

whereby i was arbitrary. Let us try and pin down the market price for a strategic market game with interim price information under $\sigma \in \Sigma^P$ in state ω analogously by simply substituting for the agents $\boldsymbol{\theta}_k(\omega)$, $k \in \{1, ..., N\}$, the new "agents"

$$[\boldsymbol{\theta}_k(\omega), \hat{\mathbf{p}}_k[\sigma_{-k}](\omega), \boldsymbol{\alpha}_k[\sigma_{-k}](\omega)], \; k \in \{1, ..., N\}.$$

In that case, (20) becomes

$$\mathbf{p}[\sigma] = \hat{\mathbf{p}}_i[\sigma_{-i}] + \boldsymbol{\alpha}_i[\sigma_{-i}] \, x_i[\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma_{-i}], \boldsymbol{\alpha}_i[\sigma_{-i}]]$$

such that

$$\hat{\mathbf{p}}_i[\sigma_{-i}] = \frac{\sum_{j \neq i} \mu[\boldsymbol{\theta}_j, \hat{\mathbf{p}}_j[\sigma_{-j}], \boldsymbol{\alpha}_j[\sigma_{-j}]] + \beta[\boldsymbol{\theta}_j, \hat{\mathbf{p}}_j[\sigma_{-j}], \boldsymbol{\alpha}_j[\sigma_{-j}]] \boldsymbol{\theta}_j + \mathbf{z}}{\sum_{j \neq i} \gamma[\boldsymbol{\theta}_j, \hat{\mathbf{p}}_j[\sigma_{-j}], \boldsymbol{\alpha}_j[\sigma_{-j}]]}, \tag{21}$$

$$\boldsymbol{\alpha}_i[\sigma_{-i}] = \frac{1}{\sum_{j \neq i} \gamma[\boldsymbol{\theta}_j, \hat{\mathbf{p}}_j[\sigma_{-j}], \boldsymbol{\alpha}_j[\sigma_{-j}]]}. \tag{22}$$

The crucial difference to the Bayesian market game is that σ_i now also appears through σ_{-j} in the the RHS of the equations (22) and (21). In other words, i's residual demand function—and thereby his interim price information—is no longer exclusively pinned down by the strategy choice σ_{-i} of his opponents as in the Bayesian market game but it might now also depend on his own strategy choice σ_i.

This, in a nutshell, is the "chicken-egg" problem of the rational expectations concept: What comes first, i's interim price information—which may depend on the choice of his strategy σ_i—or his strategy choice σ_i—which may condition on his interim price information?

5.1.3 The action-profile correspondence

I address the "chicken-egg" problem through a consistency condition which takes on the form of a system of equations. For a fixed $\sigma \in \Sigma^P$ and state ω consider the following system of $2N$

equations in the $2N$ unknown values $(\hat{p}_i, \alpha_i)$, $i = 1, .., N$:

$$\begin{aligned}
\hat{p}_1 &= \frac{\sum_{i \neq 1} \mu\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] + \beta\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] \boldsymbol{\theta}_i(\omega) + \mathbf{z}(\omega)}{\sum_{i \neq 1} \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right]} \qquad (23) \\
&\vdots \\
\hat{p}_N &= \frac{\sum_{i \neq N} \mu\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] + \beta\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] \boldsymbol{\theta}_i(\omega) + \mathbf{z}(\omega)}{\sum_{i \neq N} \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right]} \\
\alpha_1 &= \frac{1}{\sum_{i \neq 1} \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right]} \\
&\vdots \\
\alpha_N &= \frac{1}{\sum_{i \neq N} \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right]}.
\end{aligned}$$

By construction, any solution $(\hat{p}_i, \alpha_i)_{i=1,..,N}$ to (23) identifies for every trader i some interim price information $(\hat{p}_i, \alpha_i)$ that is consistent with all the other traders' interim price information $(\hat{p}_j, \alpha_j)$, $j \neq i$, in state ω under strategy-profile σ.

Collect all solutions $(\hat{p}_i, \alpha_i)_{i=1,..,N}$ to (23) in the following, possibly empty, set

$$Q^N[\sigma](\omega) \subseteq (\mathbb{R} \times \mathbb{R}_{>0})^N.$$

Next I define a correspondence which identifies for every state ω the candidate action-profiles that might be chosen by the traders' candidate agents

$$([\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i])_{i=1,..,N}$$

subject to the constraint that the interim price information of all agents $(\hat{p}_i, \alpha_i)_{i=1,..,N}$ is a solution to (23).

Definition. Action-profile correspondence. *For a fixed $\sigma \in \Sigma^P$ define the correspondence $\varphi[\sigma] : \Omega \rightrightarrows A^N$ such that*

$$\begin{aligned}
\varphi[\sigma](\omega) &= \Big\{\left(\mu\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right], \beta\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right], \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right]\right)_{i=1,..,N} \in A^N \\
&\quad \mid (\hat{p}_i, \alpha_i)_{i=1,..,N} \in Q^N[\sigma](\omega)\Big\}. \qquad (24)
\end{aligned}$$

Observe that there are three possibilities regarding the values of the action-profile correspondence:

1. If $\varphi[\sigma](\omega)$ is single-valued, the action-profile that will be chosen in state ω under strategy-profile σ is unambiguously pinned down.

2. If $\varphi[\sigma](\omega)$ is multi-valued, there are multiple candidates for the action-profile that might be chosen in state ω under σ.

3. If $\varphi[\sigma](\omega)$ is empty, there do not exist any candidates for the action-profile which might be chosen in state ω under σ.

In what follows I provide examples for all three possibilities.

Single-valued action-profile correspondence. Note that $\varphi[\sigma](\omega)$ is single-valued iff the system (23) has a unique solution, i.e., iff

$$Q^N[\sigma](\omega) = \left\{(\hat{p}_i, \alpha_i)_{i=1,..,N}\right\}.$$

I call $\sigma_i \in \Sigma_i^P$ a $(\hat{p}_i, \alpha_i)$-*constant* strategy iff

$$\sigma_i[\theta_i, \hat{p}_i, \alpha_i] = (\mu[\theta_i], \beta[\theta_i], \gamma[\theta_i])$$

for all $[\theta_i, \hat{p}_i, \alpha_i] \in \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0}$.

Lemma 1. *Suppose that all $j \neq i$ choose $(\hat{p}_j, \alpha_j)$-constant strategies $\sigma_j \in \Sigma_j^P$. Irrespective of i's strategy choice $\sigma_i \in \Sigma_i^P$, the action-profile correspondence $\varphi[\sigma] : \Omega \rightrightarrows A^N$ is single-valued in every state ω because (23) has a unique solution given as:*

$$\begin{aligned}
\hat{p}_i &= \frac{\sum_{j\neq i} \mu[\boldsymbol{\theta}_j(\omega)] + \beta[\boldsymbol{\theta}_j(\omega)]\,\boldsymbol{\theta}_i(\omega) + \mathbf{z}(\omega)}{\sum_{j\neq i} \gamma[\boldsymbol{\theta}_j(\omega)]}, \\
\alpha_i &= \frac{1}{\sum_{j\neq i} \gamma[\boldsymbol{\theta}_k(\omega)]}
\end{aligned}$$

and, for all $j \neq i$,

$$\begin{aligned}
\hat{p}_j &= \frac{\mu[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i] + \beta[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i]\,\boldsymbol{\theta}_i(\omega)}{\gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i] + \sum_{k\neq i,j} \gamma[\boldsymbol{\theta}_k(\omega)]} \\
&\quad + \frac{\sum_{k\neq i,j} (\mu[\boldsymbol{\theta}_k(\omega)] + \beta[\boldsymbol{\theta}_k(\omega)]\,\boldsymbol{\theta}_k(\omega)) + \mathbf{z}(\omega)}{\gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i] + \sum_{k\neq i,j} \gamma[\boldsymbol{\theta}_k(\omega)]}, \\
\alpha_j &= \frac{1}{\gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i] + \sum_{k\neq i,j} \gamma[\boldsymbol{\theta}_k(\omega)]}.
\end{aligned}$$

Multi– or empty valued action-profile correspondence. In general, there may exist $\sigma \in \Sigma^P$ for which $\varphi[\sigma](\omega)$ might be multi-valued or empty because the system (23) may have multiple or no solutions. By Lemma 1, this is only possible for $\sigma \in \Sigma^P$ such that there are at least two traders who do not choose $(\hat{p}_i, \alpha_i)$-constant strategies.

Example 2. Let $\pi(\mathbf{z} = 0) = 1$ and $N = 2$. Consider any $\sigma = (\sigma_1, \sigma_2)$ such that, for all i and all $[\theta_i, \hat{p}_i, \alpha_i] \in \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0}$,

$$\begin{aligned} \beta[\theta_i, \hat{p}_i, \alpha_i] &= 0, \\ \gamma[\theta_i, \hat{p}_i, \alpha_i] &= 1. \end{aligned}$$

Observe that these parameter specifications imply that both traders' interim price information only depends on the specification of their $\mu[\cdot]$-parameters because we have for every state ω

$$\begin{aligned} \hat{p}_1 &= \frac{(\mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2] + \beta[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2]\,\boldsymbol{\theta}_2(\omega))}{\gamma[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2]} = \mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2], \\ \alpha_1 &= \frac{1}{\gamma[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2]} = 1 \end{aligned}$$

as well as

$$\begin{aligned} \hat{p}_2 &= \frac{(\mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1] + \beta[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1]\,\boldsymbol{\theta}_1(\omega))}{\gamma[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1]} = \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1], \\ \alpha_2 &= \frac{1}{\gamma[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1]} = 1. \end{aligned}$$

Any solution to the system of equations (23) is thus characterized by $\hat{p}_1$ and $\hat{p}_2$ satisfying

$$\begin{aligned} \hat{p}_1 &= \mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, 1] \\ \hat{p}_2 &= \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, 1]. \end{aligned} \tag{25}$$

In what follows I make both traders' $\mu[\cdot]$-parameters dependent on their interim price information to the effect that both strategies are no longer $(\hat{p}_i, \alpha_i)$-constant.

Case (i): Multiple solutions. Further specify σ such that for all $[\theta_i, \hat{p}_i, \alpha_i] \in \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0}$,

$$\begin{aligned}
\mu[\theta_1, \hat{p}_1, \alpha_1] &= \begin{cases} 1 & \text{if } \hat{p}_1 = 0 \\ 0 & \text{else;} \end{cases} \\
\mu[\theta_2, \hat{p}_2, \alpha_2] &= \begin{cases} 2 & \text{if } \hat{p}_2 = 0 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Then there exist exactly two solutions to (25) for every $\omega \in \Omega$:

$$\begin{aligned}
\hat{p}_1 &= \mu[\boldsymbol{\theta}_2(\omega), (\hat{p}_2, 1)] = 2, \\
\hat{p}_2 &= \mu[\boldsymbol{\theta}_1(\omega), (\hat{p}_1, 1)] = 0
\end{aligned}$$

and

$$\begin{aligned}
\hat{p}_1 &= \mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, 1] = 0, \\
\hat{p}_2 &= \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, 1] = 1.
\end{aligned}$$

Consequently, for all $\omega \in \Omega$, $\varphi[\sigma](\omega)$ contains the action-profile

$$\begin{aligned}
& (\mu[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \beta[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i])_{i=1,2} \\
= \ & ((0,0,1), (2,0,1)),
\end{aligned}$$

resulting in market price $\mathbf{p}[\sigma](\omega) = 1$, as well as the action-profile

$$\begin{aligned}
& (\mu[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \beta[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i])_{i=1,2} \\
= \ & ((1,0,1), (0,0,1)),
\end{aligned}$$

resulting in market price $\mathbf{p}[\sigma](\omega) = \frac{1}{2}$. That is, the model does not unambiguously pin down which of these alternative demand-schedules—with different resulting market prices—will be submitted in any given state ω.

Case (ii): No solution. Alternatively, specify σ now such that for all $[\theta_i, \hat{p}_i, \alpha_i] \in \Theta_i \times \mathbb{R} \times \mathbb{R}_{>0}$,

$$\mu[\theta_1, \hat{p}_1, \alpha_1] = \begin{cases} 1 & \text{if } \hat{p}_1 = 0 \\ 0 & \text{else;} \end{cases}$$
$$\mu[\theta_2, \hat{p}_2, \alpha_2] = \begin{cases} 0 & \text{if } \hat{p}_2 = 0 \\ 1 & \text{else.} \end{cases}$$

Then there does not exist any solution to (25). To see this, suppose that

$$\hat{p}_2 = \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1] = 0.$$

But then

$$\mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2] = 0 \Rightarrow \hat{p}_1 = 0 \Rightarrow \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1] = 1,$$

a contradiction. Now suppose that

$$\hat{p}_2 = \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1] \neq 0,$$

implying

$$\mu[\boldsymbol{\theta}_2(\omega), \hat{p}_2, \alpha_2] = 1 \Rightarrow \hat{p}_1 = 1 \Rightarrow \mu[\boldsymbol{\theta}_1(\omega), \hat{p}_1, \alpha_1] = 0,$$

also a contradiction. Consequently, we have $\varphi[\sigma](\omega) = \emptyset$ for all ω.$\square$

5.1.4 Well-behaved strategy-profiles

In contrast to a Bayesian game Γ^B—for which every strategy profile $\sigma \in \Sigma^B$ was a $\mathcal{B}$-measurable function by construction,—we have for Γ^P that every $\sigma \in \Sigma^P$ induces an action-profile correspondence $\varphi[\sigma] : \Omega \rightrightarrows A^N$. To obtain a well-defined expected utility integral for strategy profile σ, we need to select from this correspondence a $\mathcal{B}$-measurable function $f[\sigma] : \Omega \to A^N$ satisfying $f[\sigma](\omega) \in \varphi[\sigma](\omega)$ for all ω.

If $\varphi[\sigma]$ is single-valued for all $\omega \in \Omega$, there trivially exists a $\mathcal{B}$-measurable selector iff the unique selector $f[\sigma](\omega) \in \varphi[\sigma](\omega)$ is $\mathcal{B}$-measurable. If $\varphi[\sigma]$ is empty-valued for some $\omega \in \Omega$, there cannot exist any $\mathcal{B}$-measurable selector. In case $\varphi[\sigma]$ is multi-valued for some $\omega \in \Omega$, formal conditions for the existence of a $\mathcal{B}$-measurable selector are not obvious. The Kuratowski–Ryll-Nardzewski Selection Theorem stated in Aliprantis and Border (2006, p.600) provides such sufficiency conditions for our framework. Recall that the lower inverse of $\varphi[\sigma]$ is defined as

$$\varphi^L[\sigma](C) = \{\omega \in \Omega \mid \varphi[\sigma](\omega) \cap C \neq \emptyset\} \text{ for any } C \subseteq A^N.$$

Existence of a $\mathcal{B}$-measurable selector (cf. 18.13 Kuratowski–Ryll-Nardzewski Selection Theorem in Aliprantis and Border (2006)).

- *Suppose that the correspondence $\varphi[\sigma] : \Omega \rightrightarrows A^N$ is weakly $\mathcal{B}$-measurable in the sense that $\varphi^L[\sigma](V) \in \mathcal{B}$ for all open sets V in the Euclidean product topology on A^N.*
- *Then a $\mathcal{B}$-measurable selector $f[\sigma] : \Omega \to A^N$ from $\varphi[\sigma] : \Omega \rightrightarrows A^N$ exists whenever $\varphi[\sigma](\omega)$ is non-empty and closed for all $\omega \in \Omega$.*

Example 2 continued. For Case (ii) the action-profile correspondence $\varphi[\sigma](\omega) = \emptyset$ for all ω is trivially weakly $\mathcal{B}$-measurable and closed-valued but empty for all $\omega \in \Omega$. For the multi-valued correspondence of Case (i)

$$\varphi[\sigma](\omega) = \{((0,0,1),(2,0,1)),((1,0,1),(0,0,1))\},$$

we have $\varphi^L[\sigma](V) = \Omega$ for every open V such that

$$((0,0,1),(2,0,1)) \in V \text{ or } ((1,0,1),(0,0,1)) \in V;$$

and $\varphi^L[\sigma](V) = \emptyset$ else. Because of $\emptyset, \Omega \in \mathcal{B}$, the finite-valued $\varphi[\sigma]$ is trivially weakly $\mathcal{B}$-measurable, so that there must exist some $\mathcal{B}$-measurable selector. To be precise, the set of all $\mathcal{B}$-measurable selectors for $\varphi[\sigma]$ is given as the following family

$$f_B[\sigma] : \Omega \to A^N,\ B \in \mathcal{B}$$

such that

$$f_B[\sigma](\omega) = \begin{cases} ((0,0,1),(2,0,1)) & \text{if } \omega \in B, \\ ((1,0,1),(0,0,1)) & \text{if } \omega \in \Omega \backslash B. \end{cases}$$

□

Unique Selector Assumption. *If there exist multiple $\mathcal{B}$-measurable selectors, the modeler is free to pick a unique $\mathcal{B}$-measurable selector $f^*[\sigma] : \Omega \to A^N$ with the interpretation that the value*

$$f^*[\sigma](\omega) = (\mu[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \beta[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i], \gamma[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i])_{i=1,..,N}$$

is the action-profile actually chosen by the traders in state $\omega \in \Omega$ under $\sigma \in \Sigma^P$.

The interim price information of any trader i corresponding to this unique selector $f^*[\sigma]$ is then given in state ω as

$$\begin{aligned}
\hat{\mathbf{p}}_i[\sigma](\omega) &= \frac{\sum_{j\neq i}\mu[\boldsymbol{\theta}_j(\omega), \hat{p}_j, \alpha_j] + \beta[\boldsymbol{\theta}_j(\omega), \hat{p}_j, \alpha_j]\boldsymbol{\theta}_j(\omega) + \mathbf{z}(\omega)}{\sum_{j\neq i}\gamma[\boldsymbol{\theta}_j(\omega), \hat{p}_j, \alpha_j]}, \\
\boldsymbol{\alpha}_i[\sigma](\omega) &= \frac{1}{\sum_{j\neq i}\gamma[\boldsymbol{\theta}_j(\omega), \hat{p}_j, \alpha_j]}.
\end{aligned}$$

For model-interpretational reasons we are interested in strategy-profiles for which the demand-schedule of any trader i in state ω is constant across all $\omega' \in \Omega$ such that

$$\omega' \in [\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma](\omega), \boldsymbol{\alpha}_i[\sigma](\omega)].$$

Definitions. Well-behaved strategy profiles.

(i) *I call a strategy profile $\sigma \in \Sigma^P$ well-behaved iff there exists a $\mathcal{B}$-measurable selector*

$$f^*[\sigma] = (f_1^*[\sigma], ..., f_N^*[\sigma])$$

such that $f_i^*[\sigma] : \Omega \to A$ is $\mathcal{G}(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma], \boldsymbol{\alpha}_i[\sigma])$-measurable for all i.

(ii) *I collect all well-behaved strategy profiles of Γ^P in the set $\Sigma^{P*} \subseteq \Sigma^P$.*

The upshot of this definition is that we have now pinned down for any well-behaved $\sigma \in \Sigma^{P*}$ through the selector $f^*[\sigma]$ the agents

$$([\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma](\omega), \boldsymbol{\alpha}_i[\sigma](\omega)])_{i=1,..,N}$$

that are submitting their demand-schedules in state $\omega \in \Omega$ to the Walrasian auctioneer.

5.2 Utility and Nash equilibria

Since $f^* [\sigma] : \Omega \to A^N$ is $\mathcal{B}$-measurable for every $\sigma \in \Sigma^{P*}$, we have for every well-behaved $\sigma \in \Sigma^{P*}$ a $\mathcal{B}$-measurable market price function

$$\mathbf{p}[\sigma] = \hat{\mathbf{p}}_i [\sigma] + \boldsymbol{\alpha}_i [\sigma] \, x_i [\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i [\sigma], \boldsymbol{\alpha}_i [\sigma]]$$

where i is arbitrary. The payoff-function (9) of the market scenario becomes now for $\sigma \in \Sigma^{P*}$

$$\mathbf{g}_i [\sigma] = (\mathbf{v}_i - \mathbf{p}[\sigma]) \, x_i [\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i [\sigma], \boldsymbol{\alpha}_i [\sigma]] (\mathbf{p}[\sigma]) + t \left(x_i [\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i [\sigma], \boldsymbol{\alpha}_i [\sigma]] (\mathbf{p}[\sigma]) \right)$$

whereby I set for the procurement scenario $\mathbf{g}_i [\sigma] = 0$ whenever

$$x_i [\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i [\sigma], \boldsymbol{\alpha}_i [\sigma]] (\mathbf{p}[\sigma]) > 0.$$

All traders share the same strictly increasing Bernoulli utility function such that for every well-behaved $\sigma \in \Sigma^{P*}$ the utility random variable $\mathbf{u}_i [\sigma] : \Omega \to \mathbb{R}$ with

$$\mathbf{u}_i [\sigma] (\omega) = (u \circ \mathbf{g}_i [\sigma]) (\omega) = u (\mathbf{g}_i [\sigma] (\omega))$$

is $\mathcal{B}$-measurable. I assume that $\mathbf{u}_i [\sigma]$ is integrable wrt π for every $\sigma \in \Sigma^{P*}$.

Strategy-profiles that are not well-behaved are not evaluated through some expected utility integral. Instead of providing a specific utility number for such $\sigma \notin \Sigma^{P*}$, I only assume that there exists some common upper bound for their utility.[7]

Definition. Utility.

(i) *For any well-behaved $\sigma \in \Sigma^{P*}$ the utility of trader i is given as his expected utility*

$$U_i^P (\sigma) = E (\mathbf{u}_i [\sigma]) = \int_\Omega u (\mathbf{g}_i [\sigma]) \, d\pi.$$

(ii) *For all $\sigma \notin \Sigma^{P*}$ there exists an upper bound $\bar{u} \in \mathbb{R} \cup \{-\infty\}$ such that $U_i^P (\sigma) \in \mathbb{R} \cup \{-\infty\}$ satisfies*

$$U_i^P (\sigma) \leq \bar{u}.$$

[7] If one specifies this upper bound as $\bar{u} = -\infty$, the traders would never have a strict incentive to choose some σ_i as a best response against σ_{-i} that results in $\sigma \notin \Sigma^{P*}$.

Collecting the above ingredients yields the strategic market game with interim price information

$$\Gamma^P = \left\langle (\Omega, \mathcal{B}, \pi), \Sigma_i^P, U_i^P \right\rangle_{i \in \{1,...,N\}}. \tag{26}$$

Definition. Nash equilibria. *The strategy profile*

$$\sigma^* = \left((\sigma_1^* [\theta_1, \hat{p}_1, \alpha_1])_{[\theta_1, \hat{p}_1, \alpha_1] \in \Theta_1 \times \mathbb{R} \times \mathbb{R}_{>0}}, ..., (\sigma_N^* [\theta_N, \hat{p}_N, \alpha_N])_{[\theta_N, \hat{p}_N, \alpha_N] \in \Theta_N \times \mathbb{R} \times \mathbb{R}_{>0}} \right) \in \Sigma^P$$

is a Nash equilibrium of Γ^P *iff*

$$U_i^P \left(\sigma_i^*, \sigma_{-i}^* \right) \geq U_i^P \left(\sigma_i, \sigma_{-i}^* \right) \text{ for all } \sigma_i \in \Sigma_i^P$$

for every $i \in \{1, ...N\}$.

Nash equilibria of Γ^P are not Bayesian Nash equilibria because Γ^P is not a Bayesian game. That said, a relevant class of BNE of Γ^B carries over to 'become' Nash equilibria of Γ^P.

Proposition 2.

- *Suppose that there exists a BNE* $\sigma^* \in \Sigma^B$ *of* Γ^B *with*

$$\sigma_i^* [\boldsymbol{\theta}_i] = (\mu^* [\boldsymbol{\theta}_i], \beta^* [\boldsymbol{\theta}_i], \gamma^* [\boldsymbol{\theta}_i])$$

  for all i such that σ^* *satisfies the "irrelevance of interim price information" condition (13) from Proposition 1.*

- *Then any* $\sigma^* \in \Sigma^P$ *consisting of* $(\hat{p}_i, \alpha_i)$*-constant strategies such that*

$$\sigma_i^* [\boldsymbol{\theta}_i, \hat{p}_i, \alpha_i] = (\mu^* [\boldsymbol{\theta}_i], \beta^* [\boldsymbol{\theta}_i], \gamma^* [\boldsymbol{\theta}_i]) \text{ for all } (\hat{p}_i, \alpha_i) \in \mathbb{R} \times \mathbb{R}_{>0}$$

  is a Nash equilibrium of Γ^P.

The intuition behind the formal proof of Proposition 2, based on Lemma 1, is straightforward: if all opponents $j \neq i$ chose $(\hat{p}_j, \alpha_j)$-constant strategies, i cannot influence his own interim price information through his strategy choice. If the "irrelevance of interim price information" condition from Proposition 1 holds, he would thus pick in every state of the world the same best response as in the Bayesian market game because only his private signal $\boldsymbol{\theta}_i$ matters.

6 Price-collusion in a procurement situation

Vives (2011, 2017) lists several economic situations—such as, e.g., "bidding for government procurement contracts, management consulting, or airline pricing reservations" (Vives 2011, p.1919)—in which traders compete via demand-schedules. Let me briefly illustrate why competition via linear-demand-schedules with parameters

$$\left(\mu_i\left[\cdot\right], \beta_i\left[\cdot\right], \gamma_i\left[\cdot\right]\right) \in A \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$$

is particularly relevant for procurement situations.

Example 3. Procurement situation. Suppose that the government has some noisy or noise-free liquidity demand for national electricity supply from different companies $i = 1, ..., N$ corresponding to different technologies (e.g., coal, solar, nuclear, etc.). These companies stand for the informed traders who have private signals θ_i about their respective cost functions and who observe their residual demand function with parameters $\hat{p}_i, \alpha_i$. Further suppose that the government asks the competing companies to submit bids for procurement contracts that must separately list (i) the price for the cost of building a new power plant and (ii) the subsequent per-unit price for electricity production in these new plants. If these per-unit prices are constant over the relevant production range, a bid is equivalent to the submission of a linear demand-schedule that corresponds to the linear total price function in the supplied amount x_i

$$p\left(x_i\right) = \frac{\mu_i\left[\cdot\right] + \beta_i\left[\cdot\right]\theta_i}{\gamma_i\left[\cdot\right]} - \frac{1}{\gamma_i\left[\cdot\right]} x_i.$$

Here, $\frac{\mu_i[\cdot]+\beta_i[\cdot]\theta_i}{\gamma_i[\cdot]}$ corresponds to the fixed price for building the new power plant and $\frac{1}{\gamma_i[\cdot]}$ corresponds to the per-unit price for the subsequent production of $-x_i$ units of electricity. Suppose that the companies chose some Nash equilibrium strategies which give rise to linear demand-schedules

$$x_i^*\left(p\right) = \mu_i^*\left[\cdot\right] + \beta_i^*\left[\cdot\right]\theta_i - \gamma_i^*\left[\cdot\right] p \text{ for all } p.$$

When the government learns its inelastic demand $\mathbf{z} = z > 0$, the Walrasian auctioneer determines the market clearing price p^* at which the government's demand equals the aggregate supply:

$$\sum_{i=1}^{N} x_i^* (p^*) + z = 0 \Leftrightarrow p^* = \frac{\sum_{i=1}^{N} \mu_i^* [\cdot] + \beta_i^* [\cdot] \theta_i + z}{\sum_{i=1}^{N} \gamma_i^* [\cdot]}.$$

The government pays then effectively to each company i with equilibrium supply $x_i^* (p^*) < 0$ the amount

$$p^* = p (x_i^* (p^*)) = \frac{\mu_i^* [\cdot] + \beta_i^* [\cdot] \theta_i}{\gamma_i^* [\cdot]} - \frac{1}{\gamma_i^* [\cdot]} x_i^* (p^*) .$$

Importantly, this total amount can now be split up into two installments: the government may initially pay the fixed amount $\frac{\mu_i^* [\cdot]+\beta_i^* [\cdot]\theta_i}{\gamma_i^* [\cdot]}$ for the construction of the new power plant and subsequently the per-unit price $\frac{1}{\gamma_i^* [\cdot]}$ for each of the $x_i^* (p^*) < 0$ units of electricity produced by the new power plant of i.$\square$

The following assumptions will be sufficient for establishing price-collusion between the informed traders against the liquidity trader in a procurement situation.

Assumptions. Procurement scenario.

(A1) Possible asset values: $V_i \subseteq \mathbb{R}$, $i \in \{1, ..., N\}$, *is non-empty and compact.*

(A2) Possible values for the liquidity trader's demand: $Z \subseteq \mathbb{R}_{>0}$ *is non-empty and compact.*

(A3) *There exists some* $z^* \in Z$ *such that* $\pi (\mathbf{z} = z^*) > 0$.

(A4) *The payoff-function for the procurement scenario is given as*

$$\mathbf{g}_i [\sigma] = \begin{cases} (\mathbf{v}_i - \mathbf{p} [\sigma]) \, x_i [\cdot] (\mathbf{p} [\sigma] (\omega)) + t (x_i [\cdot] (\mathbf{p} [\sigma])) & \text{if } x_i [\cdot] (\mathbf{p} [\sigma]) < 0 \\ 0 & \text{else} \end{cases}$$

such that $t : \mathbb{R} \to \mathbb{R}$ *is non-decreasing on* $(-\infty, 0]$.[8]

[8] This specification of $t(\cdot)$ covers any cost function that increases in the supplied amount $|x_i|$ for $x_i \leq 0$ such as, e.g., Vives's (2011) quadratic cost function $t (x) = -\frac{\lambda}{2} x^2$.

Denote by p_* some 'punishment' price satisfying

$$p_* \leq \min_{i \in \{1,...,N\}} \min V_i, \tag{27}$$

which exists by compactness of the V_i.

Proposition 3. *Suppose that the Assumptions (A1)-(A4) hold. Further suppose that either one of the following two conditions holds:*

C1. *The Bernoulli utility function $u(\cdot)$ is not bounded from above.*

C2. *The Bernoulli utility function $u(\cdot)$ is bounded from above and $\pi(\mathbf{z} = z^*) = 1$.*

(i) *Then there exists a sufficiently large price $p^* \in \mathbb{R}$ such that the strategy-profile $\sigma^* \in \Sigma^P$ with*

$$\begin{aligned}
\mu^*[\theta_i, \hat{p}_i, \alpha_i] &= \begin{cases} p^* - \frac{z^*}{N} & \text{if } (\hat{p}_i, \alpha_i) = \left(p^* + \frac{z^*}{N(N-1)}, \frac{1}{N-1}\right) \\ -\frac{\hat{p}_i}{\alpha_i} + p_* \left(\frac{1}{\alpha_i} + 1\right) & \text{else} \end{cases}, \\
\beta^*[\theta_i, \hat{p}_i, \alpha_i] &= 0, \\
\gamma^*[\theta_i, \hat{p}_i, \alpha_i] &= 1
\end{aligned}$$

for all $[\theta_i, \hat{p}_i, \alpha_i] \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$, $i \in \{1, ..., N\}$, is a Nash equilibrium of Γ^P.

(ii) *The corresponding equilibrium price and demand for all $i \in \{1, ..., N\}$ in state $\omega \in \Omega$ are given as*

$$\begin{aligned}
\mathbf{p}[\sigma^*](\omega) &= \begin{cases} p^* & \text{if } \mathbf{z}(\omega) = z^* \\ p_* & \text{else} \end{cases} \\
x_i^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)](\mathbf{p}[\sigma^*](\omega)) &= -\frac{\mathbf{z}(\omega)}{N}.
\end{aligned}$$

Although the formal details of the proof are complex, the basic idea is easily explained for the special case $\pi(\mathbf{z} = z^*) = 1$. In this case, the Nash equilibrium price will be p^* in every state of the world. If some trader i deviates from his equilibrium strategy σ_i^* to some σ_i resulting in a different market price than p^*, each opponent $j \neq i$ would learn through his interim price information that some deviation has happened. By construction, the opponents' Nash equilibrium strategy-profile

σ^*_{-i} results then, for any deviating σ_i, in the 'punishing' market price p_*. For sufficiently large prices p^* every trader would always rather collude than being punished.[9]

7 Concluding remarks

Strategic market games with interim price information combine (i) the rational expectations concept of interim price information with (ii) Bayesian market games in which traders receive private signals and compete via linear demand-schedules. As the main technical challenge I had to address consistency and measurability issues that arise from the "chicken-egg" problem of rational expectations. Whenever interim price information becomes relevant to the strategic choices of the traders, Nash equilibria of strategic market games with interim price information go beyond BNE for Bayesian market games. As an economically relevant application, I show that interim price information may support price-collusion between all informed traders against a liquidity trader.

[9] Obviously, such punishment of i by all $j \neq i$ would also punish i's opponents themselves, which might raise questions in analogy to perfectness criteria for Nash equilibria in sequential games. Here, I simply assume that the traders have the power to commit their agents to their chosen strategies.

Appendix: Proofs

Proof of Proposition 1. By a basic dominance property of the integral, (13) implies

$$
\begin{aligned}
& \int_\Omega E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right) d\pi \\
\geq & \int_\Omega E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right) d\pi
\end{aligned}
$$

$$\Leftrightarrow$$

$$
\begin{aligned}
& E\left(E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)\right) \\
\geq \quad & E\left(E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)\right).
\end{aligned}
$$

This is, by the law of iterated expectations (6), equivalent to

$$
\begin{aligned}
E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right]\right) & \geq E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right]\right) \\
& \Leftrightarrow \\
U_i^B\left(\sigma_i^*, \sigma_{-i}^*\right) & \geq U_i^B\left(\sigma_i, \sigma_{-i}^*\right).
\end{aligned}
$$

□□

Proof of Proposition 2. Fix σ_{-i}^* and let us check that i has no strict incentive to deviate from strategy σ_i^*. If all agents $j \neq i$ have chosen $(\hat{p}_j, \alpha_j)$-constant strategies σ_j^*, any $\left(\sigma_i, \sigma_{-i}^*\right) \in \Sigma^P$ is, by Lemma 1, well-behaved with a unique $\mathcal{B}$-measurable selector $f^*[\sigma] : \Omega \rightarrow A^N$ such that for all $\omega \in \Omega$,

$$
\begin{aligned}
\hat{\mathbf{p}}_i\left[\sigma_i, \sigma_{-i}^*\right](\omega) & = \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right](\omega) = \frac{\sum_{j \neq i} \mu\left[\boldsymbol{\theta}_j(\omega)\right] + \beta\left[\boldsymbol{\theta}_j(\omega)\right] \boldsymbol{\theta}_i(\omega) + \mathbf{z}(\omega)}{\sum_{j \neq i} \gamma\left[\boldsymbol{\theta}_j(\omega)\right]}, \\
\boldsymbol{\alpha}_i\left[\sigma_i, \sigma_{-i}^*\right](\omega) & = \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right](\omega) = \frac{1}{\sum_{j \neq i} \gamma\left[\boldsymbol{\theta}_k(\omega)\right]}.
\end{aligned}
$$

The "irrelevance of interim price information" condition from Proposition 1: for π-almost all $\omega \in \Omega$

$$
\begin{aligned}
& E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)(\omega) \\
\geq \quad & E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_{-i}^*\right]\right)\right)(\omega),
\end{aligned}
$$

implies then: for π-almost all $\omega \in \Omega$

$$\begin{aligned}
& E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_i^*, \sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_i^*, \sigma_{-i}^*\right]\right)\right)(\omega) \\
\geq \; & E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_i, \sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_i, \sigma_{-i}^*\right]\right)\right)(\omega) .
\end{aligned}$$

This yields, by a dominance property of the integral and the law of iterated expectations (6), the desired result

$$\begin{aligned}
& E\left(E\left(\mathbf{u}_i\left[\sigma_i^*, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_i^*, \sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_i^*, \sigma_{-i}^*\right]\right)\right)\right) \\
\geq \; & E\left(E\left(\mathbf{u}_i\left[\sigma_i, \sigma_{-i}^*\right] \| \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma_i, \sigma_{-i}^*\right], \boldsymbol{\alpha}_i\left[\sigma_i, \sigma_{-i}^*\right]\right)\right)\right)
\end{aligned}$$

$$\Leftrightarrow$$

$$U_i^P\left(\sigma_i^*, \sigma_{-i}^*\right) \geq U_i^P\left(\sigma_i, \sigma_{-i}^*\right).$$

□□

Proof of Proposition 3. Part 1: I determine i's expected utility from $\sigma^* \in \Sigma^P$.

Step 1. Because of $\beta^*\left[\theta_i, q_i\right] = 0$ and $\gamma^*\left[\theta_i, q_i\right] = 1$, the system of equations (23) reduces for strategy-profile σ^* to

$$\begin{aligned}
\hat{p}_1 &= \frac{\sum_{i \neq 1} \mu^*\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] + \mathbf{z}(\omega)}{N-1} \\
&\vdots \\
\hat{p}_N &= \frac{\sum_{i \neq N} \mu^*\left[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i\right] + \mathbf{z}(\omega)}{N-1}, \\
\alpha_1 &= \cdots = \alpha_N = \frac{1}{N-1}.
\end{aligned} \tag{28}$$

Consider, at first, any state ω such that $\mathbf{z}(\omega) = z^*$. Then

$$\left(\hat{p}_i, \alpha_i\right) = \left(p^* + \frac{z^*}{N(N-1)}, \frac{1}{N-1}\right) \text{ for all } i$$

is a solution to this system of equations since

$$\mu^*\left[\boldsymbol{\theta}_i(\omega), p^* + \frac{z^*}{N(N-1)}, \frac{1}{N-1}\right] = p^* - \frac{z^*}{N} \text{ for all } i$$

yields

$$\begin{aligned}
\hat{p}_i &= \frac{\sum_{j \neq i} \mu^*\left[\boldsymbol{\theta}_j(\omega), \hat{p}_j, \alpha_j\right] + z^*}{N-1} = \frac{(N-1)\left(p^* - \frac{z}{N}\right) + z^*}{N-1} \\
&= p^* + \frac{z^*}{N(N-1)}
\end{aligned}$$

for all i. Now consider any state ω such that $\mathbf{z}(\omega) \neq z^*$. Then

$$(\hat{p}_i, \alpha_i) = \left(p_* + \frac{\mathbf{z}(\omega)}{N(N-1)}, \frac{1}{(N-1)}\right) \text{ for all } i$$

is a solution to this system of equations. To see this, suppose that

$$\hat{p}_i < p_* + \frac{z^*}{N(N-1)}. \tag{29}$$

Then

$$\mu^*[\boldsymbol{\theta}_i(\omega), \hat{p}_i, \alpha_i] = -\frac{\hat{p}_i}{\alpha_i} + p_*\left(\frac{1}{\alpha_i} + 1\right) = -(N-1)\hat{p}_i + Np_* \text{ for all } i$$

implies

$$\begin{aligned}
\hat{p}_i &= \frac{(N-1)\mu^* + \mathbf{z}(\omega)}{(N-1)} = -(N-1)\hat{p}_i + Np_* + \frac{\mathbf{z}(\omega)}{N-1} \\
&\Rightarrow \\
\hat{p}_i &= p_* + \frac{\mathbf{z}(\omega)}{N(N-1)}.
\end{aligned}$$

Finally, observe that we can always find some sufficiently large p^* such that

$$\hat{p}_i = p_* + \frac{\mathbf{z}(\omega)}{N(N-1)} \leq p_* + \frac{\max Z}{N(N-1)} < p^* + \frac{z^*}{N(N-1)},$$

which confirms (29).

Step 2. Pick the selector $f^*[\sigma^*]: \Omega \to A^N$, described as the solution to (28) under Step 1, which pins down the agents

$$([\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)])_{i=1,..,N}$$

such that for every $\omega \in \Omega$ with $\mathbf{z}(\omega) = z^*$

$$\begin{aligned}
\hat{\mathbf{p}}_i[\sigma^*](\omega) &= p^* + \frac{z^*}{N(N-1)}, \\
\boldsymbol{\alpha}_i[\sigma^*](\omega) &= \frac{1}{N-1}.
\end{aligned} \tag{30}$$

whereas for every $\omega \in \Omega$ with $\mathbf{z}(\omega) \neq z^*$,

$$\begin{aligned}
\hat{\mathbf{p}}_i[\sigma^*](\omega) &= p_* + \frac{\mathbf{z}(\omega)}{N(N-1)}, \\
\boldsymbol{\alpha}_i[\sigma^*](\omega) &= \frac{1}{N-1}.
\end{aligned} \tag{31}$$

Since $f_i^*[\sigma^*]$ is $\mathcal{G}(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i[\sigma^*], \boldsymbol{\alpha}_i[\sigma^*])$-measurable, σ^* is a well-behaved strategy profile. For any $\omega \in \Omega$ with $\mathbf{z}(\omega) = z^*$ we obtain, by (30), as corresponding action

$$\begin{aligned}
\mu^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] &= p^* - \frac{z^*}{N}, \\
\beta^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] &= 0, \\
\gamma^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] &= 1,
\end{aligned}$$

for all i, yielding the market price

$$\begin{aligned}
\mathbf{p}[\sigma^*](\omega) &= \hat{\mathbf{p}}_i[\sigma^*](\omega) + \boldsymbol{\alpha}_i[\sigma^*](\omega)\left(\mu^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] - \mathbf{p}[\sigma^*](\omega)\right) \\
&= p^* + \frac{z^*}{N(N-1)} + \frac{1}{N-1}\left(\left(p^* - \frac{z^*}{N}\right) - \mathbf{p}[\sigma^*](\omega)\right) \\
&= p^* + \frac{1}{N-1}p^* - \frac{1}{N-1}\mathbf{p}[\sigma^*](\omega) \Rightarrow \mathbf{p}[\sigma^*](\omega) = p^*
\end{aligned}$$

as well as

$$\begin{aligned}
& x_i[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)](\mathbf{p}[\sigma^*](\omega)) \\
= & \left(p^* - \frac{z^*}{N}\right) - p^* = -\frac{z^*}{N}.
\end{aligned}$$

For any $\omega \in \Omega$ with $\mathbf{z}(\omega) \neq z^*$ we have, by (31),

$$\mu^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] = -\frac{\hat{\mathbf{p}}_i[\sigma^*](\omega)}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} + p_*\left(\frac{\boldsymbol{\alpha}_i[\sigma^*](\omega) + 1}{\boldsymbol{\alpha}_i[\sigma^*](\omega)}\right)$$

as well as

$$\begin{aligned}
\beta^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] &= 0, \\
\gamma^*[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)] &= 1.
\end{aligned}$$

This yields the market price

$$\begin{aligned}
\mathbf{p}[\sigma^*](\omega) &= \hat{\mathbf{p}}_i[\sigma^*](\omega) + \boldsymbol{\alpha}_i[\sigma^*](\omega)\left(-\frac{\hat{\mathbf{p}}_i[\sigma^*](\omega)}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} + p_*\left(\frac{\boldsymbol{\alpha}_i[\sigma^*](\omega) + 1}{\boldsymbol{\alpha}_i[\sigma^*](\omega)}\right) - \mathbf{p}[\sigma^*](\omega)\right) \\
&= p_* + \frac{1}{N-1}p_* - \frac{1}{N-1}\mathbf{p}[\sigma^*](\omega) \Rightarrow \mathbf{p}[\sigma^*](\omega) = p_*
\end{aligned}$$

as well as the equilibrium demand

$$\begin{aligned}
x_i[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i[\sigma^*](\omega), \boldsymbol{\alpha}_i[\sigma^*](\omega)](\mathbf{p}[\sigma^*](\omega)) &= -\frac{\hat{\mathbf{p}}_i[\sigma^*](\omega)}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} + p_*\left(\frac{1}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} + 1\right) - p_* \\
&= -\frac{\mathbf{z}(\omega)}{N}
\end{aligned}$$

because of

$$\begin{aligned}
-\frac{\hat{\mathbf{p}}_i[\sigma^*](\omega)}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} &= -\left((N-1)\left(-\frac{\hat{\mathbf{p}}_j[\sigma^*](\omega)}{\boldsymbol{\alpha}_j[\sigma^*](\omega)}+p_*\left(\frac{1}{\boldsymbol{\alpha}_j[\sigma^*](\omega)}+1\right)\right)+\mathbf{z}(\omega)\right) \\
&\Rightarrow -\frac{\hat{\mathbf{p}}_i[\sigma^*](\omega)}{\boldsymbol{\alpha}_i[\sigma^*](\omega)} = -\left((N-1)p_* + \frac{\mathbf{z}(\omega)}{N}\right).
\end{aligned}$$

Step 3. Recall that, by assumption, $\pi(\mathbf{z}=z^*)>0$. Trader i's expected utility from σ^* is then given as

$$\begin{aligned}
E(\mathbf{u}_i[\sigma^*]) &= \int_{(\mathbf{z}=z^*)} \mathbf{u}_i[\sigma^*]\,d\pi + \int_{(\mathbf{z}\neq z^*)} \mathbf{u}_i[\sigma^*]\,d\pi \\
&= \pi(\mathbf{z}=z^*)\int_{(\mathbf{z}=z^*)} u\left((\mathbf{v}_i-p^*)(-)\frac{z^*}{N}+t\left(-\frac{z^*}{N}\right)\right)d\pi(\cdot\mid\mathbf{z}=z^*) \\
&\quad +(1-\pi(\mathbf{z}=z^*))\int_{(\mathbf{z}\neq z^*)} u\left((\mathbf{v}_i-p_*)(-)\frac{\mathbf{z}}{N}+t\left(-\frac{\mathbf{z}}{N}\right)\right)d\pi(\cdot\mid(\mathbf{z}\neq z^*)).
\end{aligned}$$

Because the V_i and Z are compact and non-empty, there exists, by the Weierstrass theorem, the following minimum

$$c = \min_{(v_i,z)\in V_i\times Z}(v_i-p_*)(-)\frac{\mathbf{z}}{N}+t\left(-\frac{\max Z}{N}\right),$$

which implies

$$c \le (\mathbf{v}_i(\omega)-p_*)(-)\frac{\mathbf{z}}{N}+t\left(-\frac{\mathbf{z}(\omega)}{N}\right) \text{ for all } \omega\in(\mathbf{z}\neq z^*)$$

as $t(\cdot)$ is non-decreasing on $(-\infty,0]$. Because of $\mathbf{v}_i(\omega)\le \max V_i$, we obtain

$$\begin{aligned}
E(\mathbf{u}_i[\sigma^*]) &\ge \pi(\mathbf{z}=z^*)\,u\left((p^*-\max V_i)\frac{z^*}{N}+t\left(-\frac{z^*}{N}\right)\right) \\
&\quad +(1-\pi(\mathbf{z}=z^*))\,u(c).
\end{aligned}$$

Part 2: Let us check how i's utility is affected if he deviates from $\sigma^*=\left(\sigma_i^*,\sigma_{-i}^*\right)\in\Sigma^{P*}$ to some $\left(\sigma_i,\sigma_{-i}^*\right)\in\Sigma^P$.

Case (i). Suppose that $\left(\sigma_i,\sigma_{-i}^*\right)$ is not well-behaved, i.e., $\left(\sigma_i,\sigma_{-i}^*\right)\notin\Sigma^{P*}$. Then we have for i's utility

$$U_i^P\left(\sigma_i,\sigma_{-i}^*\right)\le\bar{u}$$

for some upper bound $\bar{u}$. If $u(\cdot)$ is not bounded from above, there exists for any strictly positive probability $\pi(\mathbf{z}=z^*)>0$ some large enough p^* such that, for all $\left(\sigma_i,\sigma^*_{-i}\right)\notin\Sigma^{P*}$,

$$\begin{aligned}\pi(\mathbf{z}=z^*)\,u\left((p^*-\max V_i)\frac{z^*}{N}+t\left(-\frac{z^*}{N}\right)\right)+(1-\pi(\mathbf{z}=z^*))\,u(c) &> \bar{u}\\ &\Rightarrow\\ E\left(\mathbf{u}_i\left[\sigma^*_i,\sigma^*_{-i}\right]\right) &> U^P_i\left(\sigma_i,\sigma^*_{-i}\right).\end{aligned}$$

If $u(\cdot)$ is bounded from above, we can always find for $\pi(\mathbf{z}=z^*)=1$ some large enough p^* such that, for all $\left(\sigma_i,\sigma^*_{-i}\right)\notin\Sigma^{P*}$,

$$\begin{aligned}(p^*-\max V_i)\frac{z^*}{N}+t\left(-\frac{z^*}{N}\right) &> 0\\ &\Leftrightarrow\\ E\left(\mathbf{u}_i\left[\sigma^*_i,\sigma^*_{-i}\right]\right) &> U^P_i\left(\sigma_i,\sigma^*_{-i}\right).\end{aligned}$$

Case (ii). Step 1. Suppose now that $\left(\sigma_i,\sigma^*_{-i}\right)$ is well-behaved but does not pin down the same agents as σ^* in some state ω. That is, we consider now the case that $\left(\sigma_i,\sigma^*_{-i}\right)\in\Sigma^{P*}$ but[10]

$$\left(\hat{\mathbf{p}}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega),\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)\right)\neq\left(p^*+\frac{z}{N(N-1)},\frac{1}{N-1}\right)\tag{32}$$

so that all $j\neq i$ choose

$$\begin{aligned}\mu^* &= -\frac{\hat{\mathbf{p}}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)}{\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)}+p_*\left(\frac{1}{\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)}+1\right),\\ \beta^* &= 0,\\ \gamma^* &= 1.\end{aligned}$$

whereby I use the abbreviations

$$\begin{aligned}\mu^* &= \mu^*\left[\boldsymbol{\theta}_j(\omega),\hat{\mathbf{p}}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega),\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)\right],\\ \beta^* &= \beta^*\left[\boldsymbol{\theta}_j(\omega),\hat{\mathbf{p}}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega),\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)\right],\\ \gamma^* &= \gamma^*\left[\boldsymbol{\theta}_j(\omega),\hat{\mathbf{p}}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega),\boldsymbol{\alpha}_j\left[\sigma_i,\sigma^*_{-i}\right](\omega)\right].\end{aligned}$$

[10] If (32) holds with equality, we would be back to Part (i) of the proof.

In what follows I will also simply write

$$\begin{aligned}\mu_i &= \mu\left[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega), \boldsymbol{\alpha}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega)\right],\\ \beta_i &= \beta\left[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega), \boldsymbol{\alpha}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega)\right],\\ \gamma_i &= \gamma\left[\boldsymbol{\theta}_i(\omega), \hat{\mathbf{p}}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega), \boldsymbol{\alpha}_i\left[\sigma_i, \sigma^*_{-i}\right](\omega)\right].\end{aligned}$$

The market price is then given as

$$\begin{aligned}\mathbf{p}\left[\sigma_i, \sigma^*_{-i}\right](\omega) &= \frac{\mu_i + \beta_i \boldsymbol{\theta}_i(\omega) + (N-1)\mu^* + \mathbf{z}(\omega)}{\gamma_i + (N-1)} \qquad (33)\\ &= \frac{\mu_i + \beta_i \boldsymbol{\theta}_i(\omega) + (N-1)\left(-\frac{\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} + p_*\left(\frac{1}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} + 1\right)\right) + \mathbf{z}(\omega)}{\gamma_i + (N-1)}.\end{aligned}$$

Next note that

$$\begin{aligned}\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega) &= \frac{\mu_i + \beta_i \theta_i + (N-2)\mu^* + \mathbf{z}(\omega)}{\gamma_i + (N-2)},\\ \boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega) &= \frac{1}{\gamma_i + (N-2)},\end{aligned}$$

implying

$$\begin{aligned}\frac{\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} &= \mu_i + \beta_i \theta_i + (N-2)\left(-\frac{\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} + p_*\left(\frac{1}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} + 1\right)\right) + \mathbf{z}(\omega)\\ &\Rightarrow\\ -\frac{\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)}{\boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)} &= -\frac{\left(\mu_i + \beta_i \theta_i + (N-2)\left(p_*\left(\gamma_i + (N-1)\right)\right) + \mathbf{z}(\omega)\right)}{N-1}. \qquad (34)\end{aligned}$$

Substituting (34) into (33) yields the market price

$$\mathbf{p}\left[\sigma_i, \sigma^*_{-i}\right](\omega) = p_*$$

for all ω for which σ_i induces the inequality (32).

Step 2. Define the event

$$\left[\sigma^* \sim \left(\sigma_i, \sigma^*_{-i}\right)\right] = \left\{\omega \in \Omega \mid \left(\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega), \boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)\right) = \left(p^* + \frac{z}{N(N-1)}, \frac{1}{N-1}\right)\right\} \in \mathcal{B}$$

and its complement

$$\left[\sigma^* \nsim \left(\sigma_i, \sigma^*_{-i}\right)\right] = \left\{\omega \in \Omega \mid \left(\hat{\mathbf{p}}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega), \boldsymbol{\alpha}_j\left[\sigma_i, \sigma^*_{-i}\right](\omega)\right) \neq \left(p^* + \frac{z}{N(N-1)}, \frac{1}{N-1}\right)\right\}.$$

Note that

$$E\left(\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]\right)=\int_{\left[\sigma^*\sim\left(\sigma_i,\sigma_{-i}^*\right)\right]}\mathbf{u}_i\left[\sigma^*\right]d\pi+\int_{\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]}\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]d\pi,$$

implying

$$\begin{array}{rcl} E\left(\mathbf{u}_i\left[\sigma^*\right]\right) & \geq & E\left(\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]\right) \\ & \Leftrightarrow & \\ \int_{\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]}\mathbf{u}_i\left[\sigma^*\right]d\pi & \geq & \int_{\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]}\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]d\pi. \end{array} \tag{35}$$

Define the event

$$\left[x_i\geq 0\right]=\left\{\omega\in\Omega\mid x_i\left[\boldsymbol{\theta}_i\left(\omega\right)\right]\left(\mathbf{p}\left[\sigma_i,\sigma_{-i}^*\right]\left(\omega\right)\right)\geq 0\right\}\in\mathcal{B}$$

and its complement

$$\left[x_i<0\right]=\left\{\omega\in\Omega\mid x_i\left[\boldsymbol{\theta}_i\left(\omega\right)\right]\left(\mathbf{p}\left[\sigma_i,\sigma_{-i}^*\right]\left(\omega\right)\right)<0\right\}.$$

Recall that we have for the procurement scenario

$$\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]\left(\omega\right)=u\left(0\right)\text{ for all }\omega\in\left[x_i\geq 0\right],$$

implying

$$\begin{aligned} & \int_{\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]}\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]d\pi \\ = & \int_{\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]\cap\left[x_i<0\right]}\mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]d\pi+\pi\left(\left[\sigma^*\nsim\left(\sigma_i,\sigma_{-i}^*\right)\right]\cap\left[x_i\geq 0\right]\right)u\left(0\right). \end{aligned}$$

Use $\mathbf{p}\left[\sigma_i,\sigma_{-i}^*\right]=p_*$ from Step 1. and observe that for $x_i\left(\omega\right)<0$

$$\begin{aligned} \mathbf{u}_i\left[\left(\sigma_i,\sigma_{-i}^*\right)\right]\left(\omega\right) &= u\left(\left(\mathbf{v}_i\left(\omega\right)-p_*\right)x_i\left(\omega\right)+t\left(x_i\left(\omega\right)\right)\right) \\ &\leq u\left(t\left(0\right)\right) \end{aligned}$$

because (i) $\mathbf{v}_i\left(\omega\right)\geq p_*$ for all ω and (ii) $t\left(x_i\left(\omega\right)\right)\leq t\left(0\right)$ for all $x_i\left(\omega\right)<0$ (Assumption A4) whereby the supplied amount of i in state ω under $\left(\sigma_i,\sigma_{-i}^*\right)$ is given as

$$x_i\left(\omega\right)=\mu_i+\beta_i\boldsymbol{\theta}_i\left(\omega\right)-\gamma_i p_*.$$

Consequently,

$$u\left(t\left(0\right)\right)\pi\left(\left[\sigma^{*}\nsim\left(\sigma_{i},\sigma_{-i}^{*}\right)\right]\cap\left[x_{i}<0\right]\right)\geq\int_{\left[\sigma^{*}\nsim\left(\sigma_{i},\sigma_{-i}^{*}\right)\right]\cap\left[x_{i}<0\right]}\mathbf{u}_{i}\left[\left(\sigma_{i},\sigma_{-i}^{*}\right)\right]d\pi.$$

Combining the above inequalities shows that we can establish (35) for any p^{*} satisfying

$$\begin{aligned}
& \pi\left(\mathbf{z}=z^{*}\right)u\left(\left(p^{*}-\max V_{i}\right)\frac{z^{*}}{N}+t\left(-\frac{z^{*}}{N}\right)\right)+\left(1-\pi\left(\mathbf{z}=z^{*}\right)\right)u\left(c\right) \\
\geq\ & \pi\left(\left[\sigma^{*}\nsim\left(\sigma_{i},\sigma_{-i}^{*}\right)\right]\cap\left[x_{i}<0\right]\right)u\left(t\left(0\right)\right)+\left(1-\pi\left(\left[\sigma^{*}\nsim\left(\sigma_{i},\sigma_{-i}^{*}\right)\right]\cap\left[x_{i}<0\right]\right)\right)u\left(0\right).
\end{aligned}$$

Such sufficiently large p^{*} exists if (i) $u\left(\cdot\right)$ is not bounded from above or (ii) if $u\left(\cdot\right)$ is bounded from above and $\pi\left(\mathbf{z}=z^{*}\right)=1$.□□

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Supplemental Appendix: Revisiting Kyle (1989) and going beyond

I illustrate how Kyle (1989) employs the "irrelevance of interim price information" condition of Proposition 1 to identify constant, symmetric BNE. Next, I slightly generalize Kyle's original distributional assumptions in order to clarify an ambiguous remark by him (Proposition 4). Finally, I show why the "irrelevance of interim price information" condition of Proposition 1 fails to establish the existence of constant, symmetric BNE if the liquidity demand is noise-free (Proposition 5).

Assumptions. Kyle's CARA-Gaussian, common value framework with noisy liquidity demand.

(A1) *The Bernoulli utility function has the constant absolute risk aversion (=CARA) form*

$$u(x) = -\exp(-\rho x), \; x \in \mathbb{R}$$

with absolute risk-aversion coefficient $\rho > 0$.

(A2) *The common asset value* $\mathbf{v}$, *the error terms* $\mathbf{e}_i = \boldsymbol{\theta}_i - \mathbf{v}$, $i = 1, ..., N$, *and the liquidity demand* $\mathbf{z}$ *are independently and normally distributed with distributional parameter values*

$$\begin{aligned} E(\mathbf{v}) &= 0, \, Var(\mathbf{v}) = \tau_v^{-1} > 0, \\ E(\mathbf{e}_i) &= 0, \, Var(\mathbf{e}_i) = \tau_e^{-1} > 0 \text{ for all } i, \\ E(\mathbf{z}) &= 0, \, Var(\mathbf{z}) = \tau_z^{-1} > 0. \end{aligned}$$

I adopt Kyle's notation and simply write $E_{\pi(\cdot|y)}(\mathbf{u})$ instead of

$$E(\mathbf{u} \parallel \mathcal{G}(\mathbf{y}))(\omega) \text{ with } \mathbf{y}(\omega) = y.$$

By Assumption A2, each random variable

$$\mathbf{g}_{x_i} = (\mathbf{v} - \hat{p}_i[\sigma_{-i}] - \alpha_i[\sigma_{-i}] x_i) x_i, \, x_i \in \mathbb{R}$$

is normally distributed under $\pi\left(\cdot \mid \theta_i, \hat{p}_i\left[\sigma_{-i}\right], \alpha_i\left[\sigma_{-i}\right]\right)$. For the log-normally distributed random variable $-\exp\left(-\rho \mathbf{g}_{x_i}\right)$ we have

$$\begin{aligned} & E_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(-\exp\left(-\rho \mathbf{g}_{x_i}\right)\right) \\ = & -\exp\left(-\rho E\left(\mathbf{g}_{x_i}\right)+\frac{1}{2}\rho^2 Var_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{g}_{x_i}\right)\right), \end{aligned}$$

which yields under Assumptions A1 and A2

$$\begin{aligned} & E_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(u\left(\mathbf{g}_{x_i}\right)\right) \\ = & -\exp -\rho\left(E_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{v}x_i\right)-\hat{p}_i\left[\sigma_{-i}\right]x_i\right. \\ & \left.-\alpha_i\left[\sigma_{-i}\right]\left(x_i\right)^2-\frac{\rho}{2}Var_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{v}\left(x_i\right)^2\right)\right). \end{aligned} \tag{1}$$

The optimal demand is uniquely pinned down by the FOC

$$\begin{aligned} 0 = & \frac{d}{dx_i}E_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{v}x_i\right)-\hat{p}_i\left[\sigma_{-i}\right]x_i \\ & -\alpha_i\left[\sigma_{-i}\right]\left(x_i\right)^2-\frac{\rho}{2}Var_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\mathbf{v}\left(x_i\right)^2\Big|_{x_i=x_i^*} \end{aligned}$$

as

$$x_i^* = \frac{E_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{v}\right)-\hat{p}_i\left[\sigma_{-i}\right]}{2\alpha_i\left[\sigma_{-i}\right]+\rho Var_{\pi(\cdot|\theta_i,\hat{p}_i[\sigma_{-i}],\alpha_i[\sigma_{-i}])}\left(\mathbf{v}\right)} \tag{2}$$

whereby the SOC holds by our assumption $\gamma_i[\cdot]>0$. Because of

$$\hat{p}_i\left[\sigma_{-i}\right] = p-\alpha_i\left[\sigma_{-i}\right]x_i^*,$$

we can equivalently rewrite (2) as

$$x_i^*\left(\theta_i,p\right) = \frac{E_{\pi(\cdot|\theta_i,p)}\left(\mathbf{v}\right)-p}{\alpha_i\left[\sigma_{-i}\right]+\rho Var_{\pi(\cdot|\theta_i,p)}\left(\mathbf{v}\right)}.$$

By Proposition 1, a constant, symmetric $\sigma^* \in \Sigma^B$ such that for all i

$$\sigma_i^*\left(\omega\right) = \left(\mu^*,\beta^*,\gamma^*\right) \text{ for all } \omega \tag{3}$$

is a BNE of Γ^B whenever the linear structure

$$x_i^*\left(\theta_i,p\right) = \mu^*+\beta^*\theta_i-\gamma^*p$$

takes on the following form:

$$\mu^*+\beta^*\theta_i-\gamma^*p = \frac{E_{\pi(\cdot|\theta_i,\mathbf{p}[\sigma^*]=p)}\left(\mathbf{v}\right)-p}{\frac{1}{(N-1)\gamma^*}+\rho Var_{\pi(\cdot|\theta_i,\mathbf{p}[\sigma^*]=p)}\left(\mathbf{v}\right)}. \tag{4}$$

Kyle's main result (cf. Theorem 5.1 and Theorem 5.2 in Kyle 1989). *Consider a Bayesian market game Γ^B satisfying the Assumptions A1 and A2 such that*

$$A = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}.$$

If the number of informed traders satisfies $N \geq 3$, then there exists a constant, symmetric BNE $\sigma^* \in \Sigma^B$ such that the functional parameter values (3) are uniquely pinned down by the following equations[1]

$$\begin{aligned} \mu^* &= 0, \qquad (5) \\ \beta^* &= \frac{(1-\varphi_I^*)\,\tau_e}{\left[(N-1)\,\gamma^*\right]^{-1}\tau_I^* + \rho} > 0, \\ \gamma^* &= \frac{\left[N\gamma^*\right]^{-1}\beta^*\tau_I^* - \varphi_I^*\tau_e}{\left[N\gamma^*\right]^{-1}\beta^*\left(\left[(N-1)\,\gamma^*\right]^{-1}\tau_I^* + \rho\right)} > 0 \end{aligned}$$

with

$$\begin{aligned} \tau_I^* &= Var^{-1}_{\pi(\cdot|\theta_i,\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) = \varphi_I^*(N-1)\,\tau_e + \tau_v + \tau_e, \qquad (6) \\ \varphi_I^* &= \frac{(\beta^*)^2(N-1)}{(\beta^*)^2(N-1) + (\tau_e)(\tau_z)^{-1}}. \qquad (7) \end{aligned}$$

Regarding the zero means in his distributional Assumption A2, Kyle (1989) makes a somewhat puzzling statement:

> "The $N+2$ random variables $\mathbf{v}, \mathbf{z}, \mathbf{e}_1, ..., \mathbf{e}_N$ are assumed to be normally and independently distributed with zero means (a normalization for $\mathbf{v}$ and the $\mathbf{e}_i$ but not for $\mathbf{z}$) [...]" (p.320).

[1] The assumption $N \geq 3$ is needed for ensuring $\gamma^* > 0$. The equations for β^* and γ^* appear under (B6) in Kyle (1989, p.346) as

$$\begin{aligned} \beta &= \frac{(1-\varphi_I)\,\tau_e}{\lambda_I\tau_I + \rho}, \\ \gamma &= \frac{\lambda\beta\tau_I - \varphi_I\tau_e}{\lambda\beta\,(\lambda_I\tau_I + \rho)}. \end{aligned}$$

I show the equivalence between Kyle's and our expression in Step 3 of the proof of Proposition 4.

I would understand if Kyle were stating that mean zero is a normalization for $\mathbf{v}$ and $\mathbf{z}$ but not for the $\mathbf{e}_i$. His original statement, however, seems to suggest that $E(\mathbf{z}) = 0$ is somehow a necessary condition. This would be strange and rather restrictive. To clarify this issue, let me generalize the specification $E(\mathbf{v}) = E(\mathbf{z}) = 0$ in Assumption A2.

Generalized Distributional Assumption.

(A2') *The common asset value* $\mathbf{v}$*, the error terms* $\mathbf{e}_i = \boldsymbol{\theta}_i - \mathbf{v}$*,* $i = 1, ..., N$*, and the liquidity demand* $\mathbf{z}$ *are independently and normally distributed with distributional parameter values*

$$\begin{aligned} E(\mathbf{v}) &\in \mathbb{R}, Var(\mathbf{v}) = \tau_v^{-1} > 0, \\ E(\mathbf{e}_i) &= 0, Var(\mathbf{e}_i) = \tau_e^{-1} > 0 \text{ for all } i, \\ E(\mathbf{z}) &\in \mathbb{R}, Var(\mathbf{z}) = \tau_z^{-1} > 0. \end{aligned}$$

Proposition 4. *Under the generalized distributional Assumption A2' the BNE parameter values (5) remain the same for* β^* *and* γ^* *but instead of* $\mu^* = 0$ *we have now*

$$\mu^* = \frac{\tau_v E(\mathbf{v}) - \left(\frac{\varphi_I^* \tau_e}{\beta^*}\right) E(\mathbf{z})}{\tau_I^* \left[(N-1)\gamma^*\right]^{-1} + \rho + \left(\frac{\varphi_I^* \tau_e}{\beta^*}\right) N}.$$

Proof of Proposition 4. As point of departure, recall the Projection Theorem: If two random variables $\mathbf{x}$ and $\mathbf{y}$ are normally distributed under π, then $\mathbf{x}$ is conditionally on $\mathbf{y}$ normally distributed with parameters

$$E_{\pi(\cdot|\mathbf{y})}(\mathbf{x}) = E(\mathbf{x}) + \frac{Cov(\mathbf{x}, \mathbf{y})}{Var(\mathbf{y})}(\mathbf{y} - E(\mathbf{y})), \tag{8}$$

$$Var_{\pi(\cdot|\mathbf{y})}(\mathbf{x}) = Var(\mathbf{x}) - \frac{Cov^2(\mathbf{x}, \mathbf{y})}{Var(\mathbf{y})}. \tag{9}$$

The Projection Theorem (cf. the proof of Step 1 of Proposition 5) yields the following characterization of any agent $\boldsymbol{\theta}_i$'s updated distributional parameters for the normal distribution of the asset value:

$$E_{\pi(\cdot|\boldsymbol{\theta}_i)}\mathbf{v} = \left(\frac{\tau_v}{\tau_v + \tau_e}\right) E(\mathbf{v}) + \frac{\tau_e}{\tau_v + \tau_e}\boldsymbol{\theta}_i, \tag{10}$$

$$Var_{\pi(\cdot|\boldsymbol{\theta}_i)}\mathbf{v} = \frac{1}{\tau_v + \tau_e}. \tag{11}$$

I proceed by proving a sequence of Steps that will culminate into Proposition 4. In what follows I fix σ^*_{-i} such that for all $j \neq i$

$$\sigma^*_j(\omega) = (\mu^*, \beta^*, \gamma^*) \text{ for all } \omega$$

and I write $\hat{\mathbf{p}}_i$ instead of $\hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right]$. For simplicity I write π_{θ_i} for the conditional probability measure $\pi(\cdot \mid \boldsymbol{\theta}_i = \theta_i)$ as well as $\pi_{\theta_i}(\cdot \mid \hat{\mathbf{p}}_i = \hat{p}_i)$ for $\pi(\cdot \mid \boldsymbol{\theta}_i = \theta_i, \hat{\mathbf{p}}_i = \hat{p}_i)$.

Step 1. *We have*

$$\begin{aligned} E_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) &= \left(1 - \frac{\varphi_I(N-1)(\tau_e)}{\tau_I}\right) E_{\pi_{\theta_i}}(\mathbf{v}) \\ &\quad + \left(\frac{\varphi_I(N-1)(\tau_e)}{\frac{\beta^*}{\gamma^*}\tau_I}\right)\left(\frac{\beta^*\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*} - \frac{E(\mathbf{z})}{(N-1)\gamma^*}\right) \end{aligned} \tag{12}$$

such that

$$\begin{aligned} \tau_I &= Var^{-1}_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) = \varphi_I(N-1)(\tau_e) + \tau_v + \tau_e, \\ \varphi_I &= \frac{(\beta^*)^2(N-1)}{(\beta^*)^2(N-1) + (\tau_e)(\tau_z)^{-1}}. \end{aligned}$$

Proof of Step 1. Under Assumption A2', the random intercept $\hat{\mathbf{p}}_i$ is normally distributed. Applying the Projection Theorem yields

$$\begin{aligned} & E_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) \\ = \;& E_{\pi_{\theta_i}}(\mathbf{v}) + \frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i)}\left(\hat{\mathbf{p}}_i - E_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i)\right) \\ = \;& E_{\pi_{\theta_i}}(\mathbf{v}) + \frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}\hat{\mathbf{p}}_i}\left(\frac{(N-1)\mu^* + \beta^*\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*} - E_{\pi_{\theta_i}}\left(\frac{(N-1)\mu^* + \beta^*\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*}\right)\right) \\ = \;& E_{\pi_{\theta_i}}(\mathbf{v}) + \frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i)}\left(\frac{\beta^*\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*} - \frac{\beta^* E_{\pi_{\theta_i}}\left(\sum_{j\neq i}\boldsymbol{\theta}_j\right) + E_{\pi_{\theta_i}}(\mathbf{z})}{(N-1)\gamma^*}\right) \\ = \;& E_{\pi_{\theta_i}}(\mathbf{v}) + \frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}\hat{\mathbf{p}}_i}\frac{\beta^*}{\gamma^*}\frac{1}{(N-1)}\left(\sum_{j\neq i}\boldsymbol{\theta}_j + \frac{\mathbf{z}}{\beta^*} - (N-1)E_{\pi_{\theta_i}}(\mathbf{v}) - \frac{E_{\pi_{\theta_i}}(\mathbf{z})}{\beta^*}\right) \\ = \;& \left(1 - \frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}\hat{\mathbf{p}}_i}\frac{\beta^*}{\gamma^*}\right) E_{\pi_{\theta_i}}(\mathbf{v}) + \left(\frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}\hat{\mathbf{p}}_i}\frac{\beta^*}{\gamma^*}\frac{1}{(N-1)}\right)\left(\sum_{j\neq i}\boldsymbol{\theta}_j + \frac{\mathbf{z}}{\beta^*} - \frac{E(\mathbf{z})}{\beta^*}\right), \end{aligned}$$

whereby I have used $E_{\pi_{\theta_i}}\left(\sum_{j\neq i}\mathbf{e}_j\right) = E\left(\sum_{j\neq i}\mathbf{e}_j\right) = 0$ as well as $E_{\pi_{\theta_i}}(\mathbf{z}) = E(\mathbf{z})$. Our independence assumptions further imply

$$
\begin{aligned}
Cov_{\pi_{\theta_i}}(\mathbf{v},\hat{\mathbf{p}}_i) &= Cov_{\pi_{\theta_i}}\left(\mathbf{v},\left(\frac{(N-1)\mu^* + \beta^*\sum_{j\neq i}\boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*}\right)\right) \\
&= Cov_{\pi_{\theta_i}}\left(\mathbf{v},\left(\frac{\mu^*}{\gamma^*} + \frac{\beta^*}{\gamma^*}\mathbf{v} + \frac{\beta^*}{(N-1)\gamma^*}\sum_{j\neq i}\mathbf{e}_j + \frac{1}{(N-1)\gamma^*}\mathbf{z}\right)\right) \\
&= Cov_{\pi_{\theta_i}}\left(\mathbf{v},\frac{\beta^*}{\gamma^*}\mathbf{v}\right) + \sum_{j\neq i}Cov_{\pi_{\theta_i}}\left(\mathbf{v},\frac{\beta^*}{(N-1)\gamma^*}\mathbf{e}_j\right) + Cov_{\pi_{\theta_i}}\left(\mathbf{v},\frac{1}{(N-1)\gamma^*}\mathbf{z}\right) \\
&= \frac{\beta^*}{\gamma^*}Var_{\pi_{\theta_i}}(\mathbf{v})
\end{aligned}
$$

as well as

$$
\begin{aligned}
Var_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i) &= Var_{\pi_{\theta_i}}\left(\frac{\mu^*}{\gamma^*} + \frac{\beta^*}{\gamma^*}\mathbf{v} + \frac{\beta^*}{(N-1)\gamma^*}\sum_{j\neq i}\mathbf{e}_j + \frac{1}{(N-1)\gamma^*}\mathbf{z}\right) \\
&= \left(\frac{\beta}{\gamma}\right)^2 Var_{\pi_{\theta_i}}\mathbf{v} + \left(\frac{\beta^*}{(N-1)\gamma^*}\right)^2\sum_{j\neq i}Var_{\pi_{\theta_i}}(\mathbf{e}_j) + \left(\frac{1}{(N-1)\gamma^*}\right)^2 Var_{\pi_{\theta_i}}(\mathbf{z}) \\
&= \left(\frac{\beta^*}{\gamma^*}\right)^2 Var_{\pi_{\theta_i}}(\mathbf{v}) + \left(\frac{\beta^*}{\gamma^*}\right)^2\frac{1}{(N-1)}Var(\mathbf{e}_j) + \left(\frac{1}{(N-1)\gamma^*}\right)^2 Var(\mathbf{z}).
\end{aligned}
$$

The conditional variance is given as

$$
\begin{aligned}
Var_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) &= Var_{\pi_{\theta_i}}(\mathbf{v}) - \frac{Cov^2_{\pi_{\theta_i}}(\mathbf{v},\hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i)} \\
&= (\tau_v+\tau_e)^{-1} - \frac{\frac{\beta^*}{\gamma^*}(\tau_v+\tau_e)^{-1}\frac{\beta^*}{\gamma^*}(\tau_v+\tau_e)^{-1}}{\left(\frac{\beta^*}{\gamma^*}\right)^2(\tau_v+\tau_e)^{-1} + \left(\frac{\beta^*}{\gamma^*}\right)^2\frac{1}{(N-1)}(\tau_e)^{-1} + \left(\frac{1}{(N-1)\gamma}\right)^2(\tau_z)^{-1}} \\
&= \frac{(\beta^*)^2(N-1) + (\tau_e)(\tau_z)^{-1}}{(\beta^*)^2(\tau_e)(N-1)^2 + (\beta^*)^2(\tau_v+\tau_e)(N-1) + (\tau_e)(\tau_z)^{-1}(\tau_v+\tau_e)}.
\end{aligned}
$$

Rewriting this in terms of the conditional precision yields

$$
\begin{aligned}
\tau_I &= Var^{-1}_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) \\
&= \frac{(\beta^*)^2(\tau_e)(N-1)^2}{(\beta^*)^2(N-1) + (\tau_e)(\tau_z)^{-1}} + \tau_v + \tau_e \\
&= \varphi_I(N-1)(\tau_e) + \tau_v + \tau_e
\end{aligned}
$$

where

$$\varphi_I = \frac{(\beta^*)^2 (N-1)}{(\beta^*)^2 (N-1) + (\tau_e)(\tau_z)^{-1}}.$$

Finally, note that

$$\begin{aligned}
\frac{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)}{Var_{\pi_{\theta_i}}(\hat{\mathbf{p}}_i)} &= \frac{Var_{\pi_{\theta_i}}(\mathbf{v}) - Var_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v})}{Cov_{\pi_{\theta_i}}(\mathbf{v}, \hat{\mathbf{p}}_i)} \\
&= \frac{\frac{1}{(\tau_v+\tau_e)} - \frac{1}{\tau_I}}{\frac{\beta^*}{\gamma^*}\frac{1}{(\tau_v+\tau_e)}} = \frac{\tau_I - (\tau_v + \tau_e)}{\frac{\beta^*}{\gamma^*}\tau_I} \\
&= \frac{\varphi_I (N-1)(\tau_e)}{\frac{\beta^*}{\gamma^*}\tau_I}
\end{aligned}$$

and substitute accordingly.□□

Step 2. *Suppose now that*

$$\sigma_i^*(\omega) = (\mu^*, \beta^*, \gamma^*) \text{ for all } \omega. \tag{13}$$

Then

$$\begin{aligned}
E_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) &= \left(\frac{\tau_v}{\tau_I}\right) E(\mathbf{v}) + (1-\varphi_I)\left(\frac{\tau_e}{\tau_I}\right)\theta_i \\
&\quad + \left(\frac{\varphi_I \tau_e}{\beta^* \tau_I}\gamma^* N\right) p - \left(\frac{\varphi_I \tau_e}{\beta^* \tau_I}\right)(N\mu^* + E(\mathbf{z})).
\end{aligned} \tag{14}$$

Proof of Step 2. Because of

$$\begin{aligned}
\hat{\mathbf{p}}_i &= \frac{(N-1)\mu^* + \beta^* \sum_{j\neq i} \boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*} \\
&\Leftrightarrow \\
\hat{\mathbf{p}}_i - \frac{\mu^*}{\gamma^*} &= \frac{\beta^* \sum_{j\neq i} \boldsymbol{\theta}_j + \mathbf{z}}{(N-1)\gamma^*},
\end{aligned}$$

we can substitute in (12) to obtain

$$\begin{aligned}
E_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) &= \left(1 - \frac{\varphi_I (N-1)(\tau_e)}{\tau_I}\right) E_{\pi_{\theta_i}}(\mathbf{v}) \\
&\quad + \left(\frac{\varphi_I (N-1)(\tau_e)}{\frac{\beta^*}{\gamma^*}\tau_I}\right)\left(\hat{\mathbf{p}}_i - \frac{\mu^*}{\gamma^*} - \frac{E(\mathbf{z})}{(N-1)\gamma^*}\right).
\end{aligned} \tag{15}$$

Use (13) to obtain

$$\begin{aligned}\hat{p}_i &= p - \frac{\mu^* + \beta^*\theta_i - \gamma^* p}{(N-1)\gamma^*} \\ &= \frac{N}{(N-1)}p - \frac{\mu^*}{(N-1)\gamma^*} - \frac{\beta^*\theta_i}{(N-1)\gamma^*}.\end{aligned}$$

Substitution in (15) yields

$$\begin{aligned}E_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i=\hat{p}_i)}(\mathbf{v}) &= \left(1 - \frac{\varphi_I (N-1)(\tau_e)}{\tau_I}\right) E_{\pi_{\theta_i}}(\mathbf{v}) \\ &+ \left(\frac{\varphi_I (N-1)(\tau_e)}{\frac{\beta^*}{\gamma^*}\tau_I}\right)\left(\frac{N}{(N-1)}p - \frac{\mu^*}{(N-1)\gamma^*} - \frac{\beta^*\theta_i}{(N-1)\gamma^*} - \frac{\mu^*}{\gamma^*} - \frac{E(\mathbf{z})}{(N-1)\gamma^*}\right).\end{aligned}$$

Observe that we have for the constant σ^* the informational equivalence

$$\mathcal{G}(\boldsymbol{\theta}_i, \mathbf{p}[\sigma^*]) = \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right]\right) = \mathcal{G}\left(\boldsymbol{\theta}_i, \hat{\mathbf{p}}_i\left[\sigma^*_{-i}\right], \boldsymbol{\alpha}_i\left[\sigma^*_{-i}\right]\right)$$

so that

$$E_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) = E_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i=\hat{p}_i)}(\mathbf{v}).$$

By (10), we have

$$E_{\pi_{\theta_i}}(\mathbf{v}) = \left(\frac{\tau_v}{\tau_v+\tau_e}\right) E(\mathbf{v}) + \frac{\tau_e}{\tau_v+\tau_e}\theta_i.$$

Substituting and rearranging terms yields

$$\begin{aligned}E_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) &= \left(1 - \frac{\varphi_I (N-1)(\tau_e)}{\tau_I}\right) E(\mathbf{v}) \\ &+ \left[\left(1 - \frac{\varphi_I (N-1)\tau_e}{\tau_I}\right)\left(\frac{\tau_e}{\tau_v+\tau_e}\right) - \left(\frac{\varphi_I\tau_e}{\tau_I}\right)\right]\theta_i \\ &+ \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right)\gamma^* N p - \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right)(N\mu^* + E(\mathbf{z})),\end{aligned}$$

which can be further transformed into (14).□□

Step 3. *The structural equation (4) holds iff the functional parameter values satisfy the following three equations*

$$\begin{aligned}\mu^* &= \frac{\left(\frac{\tau_v}{\tau_I}\right) E(\mathbf{v}) - \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right) E(\mathbf{z})}{\left(\left[(N-1)\gamma^*\right]^{-1} + \rho\tau_I^{-1}\right) + \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right) N}, \\ \beta^* &= \frac{(1-\varphi_I)\tau_e}{\lambda_I\tau_I + \rho}, \\ \gamma^* &= \frac{\lambda\beta^*\tau_I - \varphi_I\tau_e}{\lambda\beta^*(\lambda_I\tau_I + \rho)},\end{aligned}$$

with

$$\begin{aligned}
\tau_I &= Var^{-1}_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*])}(\mathbf{v}) = Var^{-1}_{\pi_{\theta_i}(\cdot|\hat{\mathbf{p}}_i)}(\mathbf{v}) = \varphi_I(N-1)(\tau_e) + \tau_v + \tau_e, \\
\varphi_I &= \frac{(\beta^*)^2(N-1)}{(\beta^*)^2(N-1) + (\tau_e)(\tau_z)^{-1}}.
\end{aligned}$$

and

$$\begin{aligned}
\lambda_I &= [(N-1)\gamma^*]^{-1}, \\
\lambda &= [N\gamma^*]^{-1}.
\end{aligned}$$

Proof of Step 3. Rewrite the structural equation (4) as

$$\mu^* + \beta^*\theta_i - \gamma^* p = \frac{E_{\pi(\cdot|\theta_i,\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) - p}{\lambda_I + \rho\tau_I^{-1}}.$$

Substituting (14) in the right hand side yields

$$\begin{aligned}
& \frac{E_{\pi(\cdot|\theta_i,\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) - p}{\lambda_I + \rho\tau_I^{-1}} \\
= & \frac{1}{\lambda_I + \rho\tau_I^{-1}}\left(\left(\frac{\tau_v}{\tau_I}\right)E(\mathbf{v}) - \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right)(N\mu^* + E(\mathbf{z}))\right) \\
& + \frac{1}{\lambda_I + \rho\tau_I^{-1}}\left[(1-\varphi_I)\left(\frac{\tau_e}{\tau_I}\right)\right]\theta_i \\
& + \frac{1}{\lambda_I + \rho\tau_I^{-1}}\left[\frac{\varphi_I\tau_e}{\beta^*\tau_I}\gamma^* N - 1\right]p.
\end{aligned}$$

Consequently, the μ^*, β^*, γ^* must satisfy the equations

$$\begin{aligned}
\mu^* &= \frac{1}{\lambda_I + \rho\tau_I^{-1}}\left(\left(\frac{\tau_v}{\tau_I}\right)E(\mathbf{v}) - \left(\frac{\varphi_I\tau_e}{\beta^*\tau_I}\right)(N\mu^* + E(\mathbf{z}))\right), \\
\beta^* &= \frac{1}{\lambda_I + \rho\tau_I^{-1}}\left[(1-\varphi_I)\left(\frac{\tau_e}{\tau_I}\right)\right]\theta_i, \\
\gamma^* &= -\frac{1}{\lambda_I + \rho\tau_I^{-1}}\left[\frac{\varphi_I\tau_e}{\beta^*\tau_I}\gamma^* N - 1\right].
\end{aligned}$$

Rearranging yields the desired expressions

$$\begin{aligned}
\mu^* &= \frac{\left(\frac{\tau_v}{\tau_I}\right) E(\mathbf{v}) - \left(\frac{\varphi_I \tau_e}{\beta^* \tau_I}\right) E(\mathbf{z})}{\left(\left[(N-1)\gamma^*\right]^{-1} + \rho\tau_I^{-1}\right) + \left(\frac{\varphi_I \tau_e}{\beta^* \tau_I}\right) N}, \\
\beta^* &= \frac{(1-\varphi_I)\tau_e}{\lambda_I \tau_I + \rho}, \\
\gamma^* &= \frac{1 - \frac{\varphi_I \tau_e}{\beta^* \tau_I}(\lambda)^{-1}}{\lambda_I + \rho\tau_I^{-1}} = \frac{\frac{\lambda\beta^*\tau_I - \varphi_I\tau_e}{\lambda\beta^*\tau_I}}{\frac{\lambda_I\tau_I+\rho}{\tau_I}} = \frac{\lambda\beta^*\tau_I - \varphi_I\tau_e}{\lambda\beta^*(\lambda_I\tau_I+\rho)},
\end{aligned}$$

which make the above analysis comparable to Kyle's original analysis. In particular, observe that we have recovered Kyle's original characterizations of the parameter values β^*, γ^* while the distributional assumption $E(\mathbf{v}) = E(\mathbf{z}) = 0$ recovers Kyle's original characterization $\mu^* = 0$.□□

The next proposition implies that Kyle's analytical approach—based on the "irrelevance of interim price information" condition of Proposition 1—cannot establish the existence of constant, symmetric BNE if the liquidity demand is noise-free.

Proposition 5. *Suppose that the Assumptions A1 and A2' hold with the qualification that $\pi(\mathbf{z} = z) = 1$ for some $z \in \mathbb{R}$. Then there does not exist any constant, symmetric $\sigma^* \in \Sigma^B$ for which the structural equation (4) holds.*

Proof of Proposition 5.

Step 1. *The common asset value $\mathbf{v}$ under any conditional probability measure $\pi_{\boldsymbol{\theta}_i}\left(\cdot \mid \sum_{k=1}^N \boldsymbol{\theta}_k\right)$ is normally distributed with distributional values*

$$\begin{aligned}
E_{\pi_{\boldsymbol{\theta}_i}\left(\cdot\mid\sum_{k=1}^N \boldsymbol{\theta}_k\right)}(\mathbf{v}) &= \left(\frac{\tau_v}{\tau_v + N\tau_e}\right) E(\mathbf{v}) + \frac{\tau_e}{\tau_v + N\tau_e}\left(\sum_{k=1}^N \boldsymbol{\theta}_k\right), \qquad (16) \\
Var_{\pi_{\boldsymbol{\theta}_i}\left(\cdot\mid\sum_{k=1}^N \boldsymbol{\theta}_k\right)}(\mathbf{v}) &= \frac{1}{N\tau_e + \tau_v}.
\end{aligned}$$

Proof of Step 1. For the normally distributed $\mathbf{v}$ the sample mean $\sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}$ is a 'sufficient statistic' for the data sample $(\boldsymbol{\theta}_1, .., \boldsymbol{\theta}_n)$ and therefore also for $\left(\boldsymbol{\theta}_i, \sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}\right)$. Consequently, we have

$$\begin{aligned}
E_{\pi_{\boldsymbol{\theta}_i}\left(\cdot\mid\sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}) &= E_{\pi\left(\cdot\mid\sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}), \\
Var_{\pi_{\boldsymbol{\theta}_i}\left(\cdot\mid\sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}) &= Var_{\pi\left(\cdot\mid\sum_{k=1}^N \frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}).
\end{aligned}$$

Next observe that $\sum_{k=1}^{N} \frac{\boldsymbol{\theta}_k}{N} = \mathbf{v} + \sum_{k=1}^{N} \frac{\mathbf{e}_k}{N}$ and $\sum_{k=1}^{N} \boldsymbol{\theta}_k$ are informationally equivalent. Let $\mathbf{x} = \mathbf{v}$ and $\mathbf{y} = \mathbf{v} + \sum_{k=1}^{N} \frac{\mathbf{e}_k}{N}$ in the Projection Theorem to obtain

$$\begin{aligned}
E_{\pi_{\boldsymbol{\theta}_i}\left(\cdot|\sum_{k=1}^{N}\boldsymbol{\theta}_k\right)}(\mathbf{v}) &= E_{\pi\left(\cdot|\sum_{k=1}^{N}\frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}) \\
&= E(\mathbf{v}) + \frac{Cov\left(\mathbf{v},\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}{Var\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N} - E\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)\right) \\
&= \left(1 - \frac{Cov\left(\mathbf{v},\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}{Var\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}\right)E(\mathbf{v}) + \frac{Cov\left(\mathbf{v},\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}{Var\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)}\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right).
\end{aligned}$$

Substituting

$$\begin{aligned}
Cov\left(\mathbf{v},\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right) &= Cov(\mathbf{v},\mathbf{v}) + \sum_{k=1}^{N} Cov\left(\mathbf{v},\frac{\mathbf{e}_k}{N}\right) = Var(\mathbf{v}), \\
Var\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right) &= Var(\mathbf{v}) + \sum_{k=1}^{N} Var\left(\frac{\mathbf{e}_k}{N}\right) = Var(\mathbf{v}) + \frac{1}{N}Var(\mathbf{e}_k),
\end{aligned}$$

and rearranging yields the desired expression (16). For the variance we have

$$\begin{aligned}
Var_{\pi\left(\cdot|\sum_{k=1}^{N}\boldsymbol{\theta}_k\right)}(\mathbf{v}) &= Var_{\pi\left(\cdot|\sum_{k=1}^{N}\frac{\boldsymbol{\theta}_k}{N}\right)}(\mathbf{v}) \\
&= Var(\mathbf{v}) - \frac{Cov\left(\mathbf{v},\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)^2}{Var\left(\mathbf{v}+\sum_{k=1}^{N}\frac{\mathbf{e}_k}{N}\right)} \\
&= \tau_v^{-1} - \frac{\left(\tau_v^{-1}\right)^2}{\tau_v^{-1} + \frac{1}{N}\tau_e^{-1}} = \frac{\tau_v^{-1}\frac{1}{N}\tau_e^{-1}}{\tau_v^{-1} + \frac{1}{N}\tau_e^{-1}} \\
&= \frac{1}{N\tau_e + \tau_v}.
\end{aligned}$$

□□

Step 2. *For any constant, symmetric $\sigma^* \in \Sigma^B$ with*

$$\sigma_i^*(\omega) = (\mu^*, \beta^*, \gamma^*) \text{ for all } \omega \text{ and } i$$

we have

$$E_{\pi_{\boldsymbol{\theta}_i}(\cdot|\mathbf{p}[\sigma^*])}(\mathbf{v}) = \left(\frac{\tau_v}{\tau_v + N\tau_e}\right) E(\mathbf{v}) + \frac{\tau_e}{\tau_v + N\tau_e}\left(\frac{N\gamma^*\mathbf{p}[\sigma^*] - N\mu^* - z}{\beta^*}\right),$$
$$Var_{\pi_{\boldsymbol{\theta}_i}(\cdot|\mathbf{p}[\sigma^*])}(\mathbf{v}) = \frac{1}{N\tau_e + \tau_v}.$$

Proof of Step 2. Due to our assumption of a noise-free liquidity demand we have informational equivalence between $\mathbf{p}[\sigma^*]$ and $\sum_{i=1}^{N}\boldsymbol{\theta}_i$ by the one-to-one mapping

$$\mathbf{p}[\sigma^*] = \frac{N\mu^* + \beta^*\sum_{k=1}^{N}\boldsymbol{\theta}_k + z}{N\gamma^*}$$
$$\Leftrightarrow$$
$$\frac{N\gamma^*\mathbf{p}[\sigma^*] - N\mu^* - z}{\beta^*} = \sum_{k=1}^{N}\boldsymbol{\theta}_k. \tag{17}$$

Substituting (17) in (16) yields the desired result. The same argument applies to the variance.□□

Step 3. *There do not exist any parameter values*

$$(\mu^*, \beta^*, \gamma^*) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$$

which satisfy the structural equation

$$\mu^* + \beta^*\theta_i - \gamma^* p = \frac{E_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*]=p)}(\mathbf{v}) - p}{\frac{1}{(N-1)\gamma^*} + \rho Var_{\pi_{\theta_i}(\cdot|\mathbf{p}[\sigma^*]=p)}(\mathbf{v})}. \tag{18}$$

Proof of Step 3. By Step 2, (18) becomes

$$\mu^* + \beta^*\theta_i - \gamma^* p \tag{19}$$
$$= \frac{\left(\frac{\tau_v}{\tau_v+N\tau_e}\right)E(\mathbf{v}) + \frac{\tau_e}{\tau_v+N\tau_e}\left(\frac{N\gamma^*p - N\mu^* - z}{\beta^*}\right)}{\frac{1}{(N-1)\gamma^*} + \rho\frac{1}{N\tau_e+\tau_v}}.$$

Since θ_i does not appear on the RHS of the equation (19), we must have $\beta^* = 0$. However, for $\beta^* = 0$ the RHS of the equation is not well-defined as we have to divide by zero. This concludes the proof of Proposition 5 because the structural equation (18) is necessary for the "irrelevance of interim price information" condition of Proposition 1.□□

The intuitive reason why we could not identify any constant, symmetric BNE for Kyle's model with noise-free liquidity demand had been already formulated by Vives (2011) as follows:

“The reason should be well understood: If the price reveals the common value, then no seller has an incentive to put any weight on his signal (and the incentives to acquire information disappear as well). But if sellers put no weight on their signals, then the price cannot contain any information on the costs parameters. This is the essence of the Grossman-Stiglitz paradox (1980).” (p.1927)